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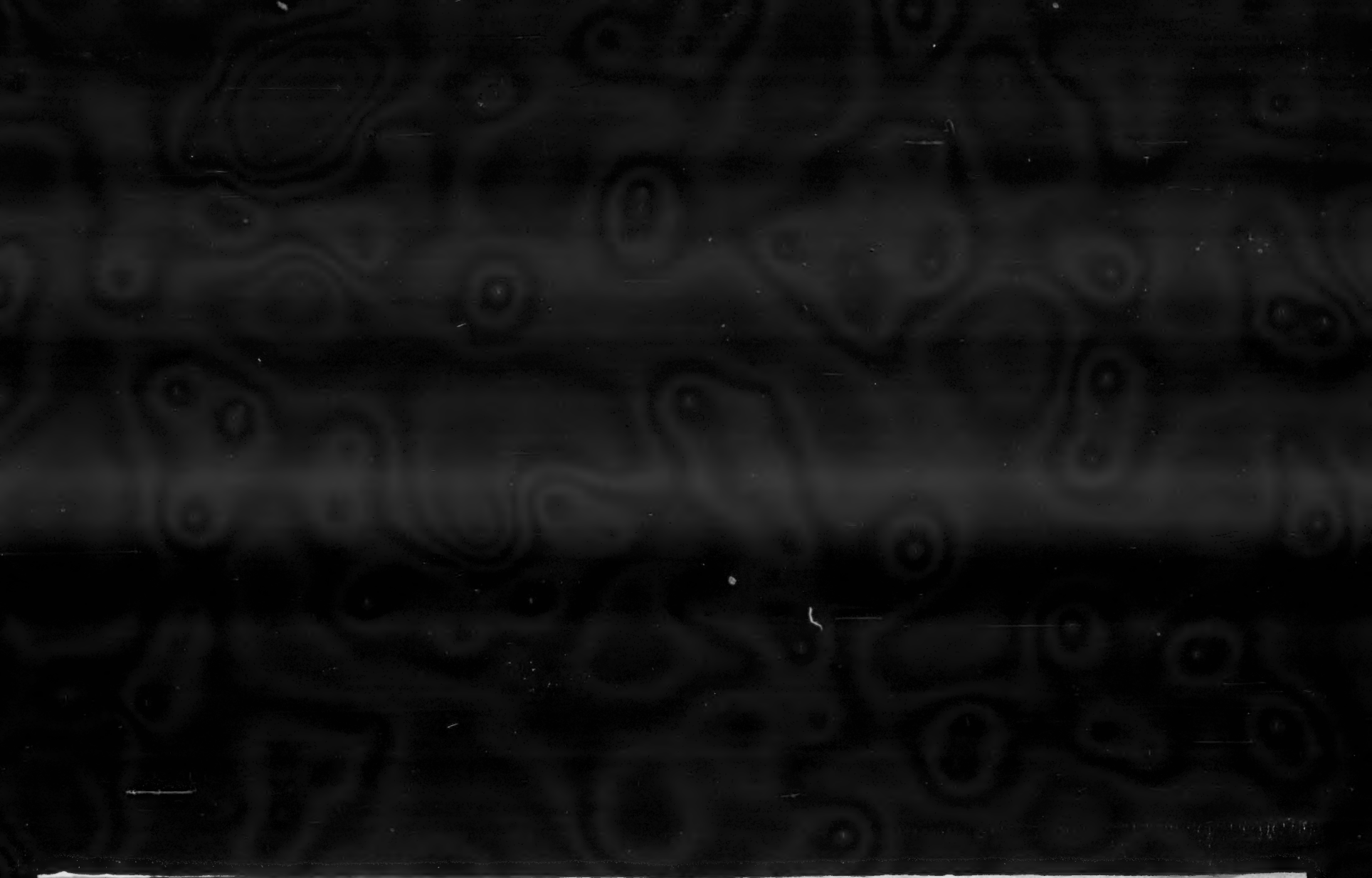

THE FAILURE OF THE CLIFFORD CHAIN.

By Walter B. Carver.

The Clifford chain theorem* defines, for a set of n lines in a plane no two of which are parallel, a Clifford circle when n is odd and a Clifford point when n is even. The Clifford point for two lines is their point of intersection, and the Clifford circle for three lines is the circum-circle. For any odd n, each set of $n-1$ lines out of the n lines determines a Clifford point, and the n such Clifford points lie on a circle, the Clifford circle of the n lines. For any even n (greater than 2), each set of $n-1$ lines determines a Clifford circle, and the n such Clifford circles pass through a point, the Clifford point of the n lines. In his proof of the theorem, Clifford does not raise the question of the existence of exceptional sets of lines for which the theorem may fail.

Kantor pointed out† that when five lines touch a deltoid (a hypocycloid of three cusps) the five Clifford points of the sets of four lines lie on a straight line instead of a circle; and that for six lines touching a deltoid, the six Clifford *lines* for the sets of five lines are not concurrent but are tangents of a new deltoid.

In his paper "On the Metric Geometry of the Plane N-Line,"‡ Morley gives, incidentally, an analytic proof of Clifford's theorem; shows that any odd number of lines (greater than three) determine, when they satisfy a certain analytic condition, a Clifford *line* rather than a circle; and intimates that further degeneracy may occur.

It is the purpose of the present paper to consider all the possibilities of failure of the Clifford chain, and to examine the conditions on a set of lines which cause such failure.

§ 1. *Characteristic Constants; Map and Envelope Equations.*

The analysis used will be the circular coördinates and characteristic constants of Morley's paper. It will be necessary to give a few preliminary definitions, and to state certain fundamental relations to be used later. These relations will be stated without proof, as some of them may be found

* Clifford, "Synthetic Proof of Miquel's Theorem," *Messenger of Mathematics*, Vol. 5, page 124; 1870.

† Kantor, "Die Tangentengeometrie an der Steiner'schen Hypocycloid," *Wiener Sitzungsberichte*, Vol. 78, page 204; 1878.

‡ *Transactions of the Amer. Math. Soc.*, Vol. 1, page 97; 1900.

in Morley's paper and the others may be verified by the reader without difficulty.

The circular coördinates x and y of a point are defined by the equations $x = X + iY$, $y = X - iY$, where X, Y are rectangular Cartesian coördinates and i is the imaginary unit. A complex number whose absolute value is unity will be called a turn. A real line is represented by a linear equation $tx + y = ct$, where c is any complex number and t a turn such that $\bar{c} = ct$. The point* c is the reflexion of the origin in the line (briefly, the reflex point of the line), and the turn t gives the inclination of the line and will be called the clinant. Two lines with clinants t_1 and t_2 are parallel if $t_1 = t_2$, and perpendicular if $t_1 = -t_2$.

A set of n lines (no two of which are parallel) determine uniquely a set of n characteristic constants, defined as follows:

$$a_i = \frac{1}{T}\begin{vmatrix} c_1 t_1^{n-i} & t_1^{n-2} & t_1^{n-3} & \cdots & t_1 & 1 \\ c_2 t_2^{n-i} & t_2^{n-2} & t_2^{n-3} & \cdots & t_2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_n t_n^{n-i} & t_n^{n-2} & t_n^{n-3} & \cdots & t_n & 1 \end{vmatrix}, \qquad i = 1, 2, \cdots, n,$$

where

$$T = \begin{vmatrix} t_1^{n-1} & t_1^{n-2} & \cdots & t_1 & 1 \\ t_2^{n-1} & t_2^{n-2} & \cdots & t_2 & 1 \\ \cdots & & & & \\ t_n^{n-1} & t_n^{n-2} & \cdots & t_n & 1 \end{vmatrix}.$$

Between these n a's we have the relations

$$\bar{a}_i = (-1)^{n-1} S_n a_{n-i+1}, \qquad i = 1, 2, \cdots, n,$$

where $S_n = t_1 t_2 t_3 \cdots t_n$. Since $S_n \neq 0$, a_i and a_{n-i+1} (called complementary a's) vanish together.

To fix an arbitrary set of n lines,

(1) for n even, $a_1, a_2, \cdots, a_{n/2}$ and the n clinants $t_1, t_2, \cdots, t_n$ (no two of them equal) may be chosen arbitrarily; and the remaining a's and the set of n lines are then uniquely fixed:

(2) for n odd, $a_1, a_2, \cdots, a_{(n+1)/2}$, and any $n-1$ of the n clinants $t_1, t_2, \cdots, t_n$ may be chosen arbitrarily; and the remaining t and a's and the set of n lines are then uniquely fixed.

If one of a set of n lines is omitted, the constants for the remaining set of $n - 1$ lines are

$$\alpha_i = a_i - t a_{i+1}, \qquad i = 1, 2, \cdots, n - 1,$$

* The phrase "the point c" is used in the sense of the usual relation between the complex numbers and the points of a plane. The point c is the *real* point whose circular coördinates are $(c, \bar{c})$.

where t is the clinant of the omitted line. More generally, if r lines are omitted, the constants for the $n - r$ lines are

$$\alpha_i = a_i - S_1 a_{i+1} + S_2 a_{i+2} - \cdots + (-1)^r S_r a_{i+r}, \qquad i = 1, 2, \cdots, n - r,$$

where the S's are the ordinary symmetric functions* of the r clinants of the omitted lines.

A set of n lines are concurrent if, and only if,

$$a_2 = a_3 = \cdots = a_{n-1} = 0;$$

and the lines will then meet at the point a_1. If $n - r$ lines out of a set of n lines are concurrent, we must then have

$$\begin{aligned}
&a_2 - S_1 a_3 + S_2 a_4 - \cdots + (-1)^r S_r a_{r+2} = 0,\\
&a_3 - S_1 a_4 + S_2 a_5 - \cdots + (-1)^r S_r a_{r+3} = 0,\\
&\vdots\\
&a_{n-r-1} - S_1 a_{n-r} - \cdots + (-1)^r S_r a_{n-1} = 0,
\end{aligned}$$

and the $n - r$ lines meet at the point

$$a_1 - S_1 a_2 + \cdots + (-1)^r S_r a_{r+1};$$

where the S's are for the clinants of the omitted lines. If $n - r \leqq (n/2) + 1$, this imposes no condition on the a's alone; but if $n - r > (n/2) + 1$, the matrix

$$\begin{Vmatrix}
a_2 & a_3 & \cdots & a_{r+2}\\
a_3 & a_4 & \cdots & a_{r+3}\\
\cdot & \cdot & \cdot & \cdot\\
a_{n-r-1} & a_{n-r} & \cdots & a_{n-1}
\end{Vmatrix}$$

must be of rank r if $n - r$ (and no more) of the n lines are concurrent.

In the following sections an important rôle will be played by determinants of the form

$$\begin{vmatrix}
a_i & a_{i+1} & \cdots & a_{i+k-1}\\
a_{i+1} & a_{i+2} & \cdots & a_{i+k}\\
\cdot & \cdot & \cdot & \cdot\\
a_{i+k-1} & a_{i+k} & \cdots & a_{i+2k-2}
\end{vmatrix}$$

where the elements are the characteristic constants of a set of n lines. Such a determinant of the kth order is equal to a determinant of the nth order in the reflex points and clinants of the n lines, as follows:

$$\begin{vmatrix}
a_i & a_{i+1} & \cdots & a_{i+k-1}\\
a_{i+1} & a_{i+2} & \cdots & a_{i+k}\\
\cdot & \cdot & \cdot & \cdot\\
a_{i+k-1} & a_{i+k} & \cdots & a_{i+2k-2}
\end{vmatrix}$$

* Throughout the paper S_i, $i = 1, 2, \cdots k$, will represent the ordinary symmetric functions of k turns $t_1, t_2, \cdots t_k$; *i.e.*, the S's will be the coefficients of the equation $t^k - S_1 t^{k-1} + S_2 t^{k-2} - S_3 t^{k-3} + \cdots + (-1)^k S_k = 0$ whose roots are the k turns.

$$= \frac{(-1)^{k(k-1)/2}}{T} \begin{vmatrix} c_1 t_1^{n-i-k+1} & c_1 t_1^{n-i-k} & \cdots & c_1 t_1^{n-i-2k+2} & t_1^{n-k-1} & t_1^{n-k-2} & \cdots & t_1 & 1 \\ c_2 t_2^{n-i-k+1} & c_2 t_2^{n-i-k} & \cdots & c_2 t_2^{n-i-2k+2} & t_2^{n-k-1} & t_2^{n-k-2} & \cdots & t_2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_n t_n^{n-i-k+1} & c_n t_n^{n-i-k} & \cdots & c_n t_n^{n-1-2k+2} & t_n^{n-k-1} & & \cdots & t_n & 1 \end{vmatrix}$$

where

$$T = \begin{vmatrix} t_1^{n-1} & t_1^{n-2} & \cdots & t_1 & 1 \\ t_2^{n-1} & t_2^{n-2} & \cdots & t_2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ t_n^{n-1} & t_n^{n-2} & \cdots & t_n & 1 \end{vmatrix}.$$

Also such a determinant in the characteristic constants (α's) for $n-1$ out of n lines may be expressed in terms of the characteristic constants (a's) for the n lines thus

$$\begin{vmatrix} \alpha_i & \alpha_{i+1} & \cdots & \alpha_{i+k-1} \\ \alpha_{i+1} & \alpha_{i+2} & \cdots & \alpha_{i+k} \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{i+k-1} & \alpha_{i+k} & \cdots & \alpha_{i+2k-2} \end{vmatrix} = \begin{vmatrix} a_i & a_{i+1} & \cdots & a_{i+k-1} & a_{i+k} \\ a_{i+1} & a_{i+2} & \cdots & a_{i+k} & a_{i+k+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{i+k-1} & a_{i+k} & \cdots & a_{i+2k-2} & a_{i+2k-1} \\ t^k & t^{k-1} & \cdots & t & 1 \end{vmatrix}$$

where t is the clinant of the omitted line.

The determinants*

$$\begin{vmatrix} a_i & \cdots & a_{i+k-1} \\ \cdot & \cdot & \cdot \\ a_{i+k-1} & \cdots & a_{i+2k-2} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_{n-i-2k+3} & \cdots & a_{n-i-k+2} \\ \cdot & \cdot & \cdot \\ a_{n-i-k+2} & \cdots & a_{n-i+1} \end{vmatrix}$$

will be called complementary. Two such complementary determinants must obviously vanish together.

The rational curves which occur in the paper will be represented either by "map" equations or by "envelope" equations. The map equation

$$x = R(t)$$

expresses x as a rational function of a turn parameter t. This equation always implies the corresponding equation

$$y = \bar{R}(1/t),$$

where $\bar{R}$ means the function obtained by replacing the coefficients in the function R by their conjugates. When t runs through turn values, x gives the *real* points of the curve and x and y are conjugates. If t is given values other than turn values, x and y will not (in general) be conjugates, but will be the circular coördinates of *imaginary* points of the curve.

* Determinants of this type will be sufficiently indicated by giving only the four corner elements.

The following special map equations may be noted:

$$x = c + kt$$

is a circle with center at c and radius $|\, k \,|$.

$$x = c + \frac{k}{(1 + kt/\bar{k})^2}$$

is a parabola of which the point c is the focus and k represents a vector from the focus to its reflexion in the directrix. In this form the parameter t is the clinant of the tangent to the curve at the point x.

$$x = c + k\left(\frac{\bar{k}^2}{k^2t^2} - \frac{2kt}{\bar{k}}\right)$$

is a deltoid of which c is the center and $3k$ represents a vector from the center to one of the cusps. In this case also t is the clinant of the tangent at the point x.

The envelope equation of a curve is of the form

$$tx + y = R(t),$$

where R is a rational function satisfying the identical relation

$$\bar{R}(1/t) = R(t)/t.$$

For a given turn value of t the envelope equation represents a straight line, and as t runs through turn values this line envelopes the curve. The map equation of the curve is obtained at once from this envelope equation by partial differentiation with respect to t, giving

$$x = R'(t),$$

where the parameter t is the clinant of the tangent at the point x.

The envelope equation of a parabola is

$$tx + y = (\alpha t^2 + rt + \bar{\alpha})/(\gamma t + \bar{\gamma}),$$

where r is real, $\gamma \neq 0$, and the denominator is not a factor of the numerator. Similarly, the envelope equation of a deltoid is

$$tx + y = (\alpha_1 t^3 + \alpha_2 t^2 + \bar{\alpha}_2 t + \bar{\alpha}_1)/rt,$$

where r is real and $\neq 0$, and $\alpha_1 \neq 0$. Differentiating these equations with respect to t, and identifying the results with the map equations of these same curves, we see that the focus of the parabola is at the point α/γ; while the center of the deltoid is at α_2/r, and its size is given by $|\, \alpha_1 \,|/r$.

The most general curve which can be represented by an envelope equation is treated in sections 3 and 8.

§ 2. *Improper Sets of* 3, 4, 5, *and* 6 *Lines.*

The map equation of the Clifford circle as given by Morley is, for 3 lines,

$$x = a_1 - a_2 t,$$

and for $2p - 1$ lines, where $p \geqq 3$,

$$x = \frac{\begin{vmatrix} a_1 & \cdots & a_{p-1} \\ \cdot & \cdot & \cdot \\ a_{p-1} & \cdots & a_{2p-3} \end{vmatrix}}{\begin{vmatrix} a_3 & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-3} \end{vmatrix}} - \frac{\begin{vmatrix} a_2 & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-2} \end{vmatrix}}{\begin{vmatrix} a_3 & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-3} \end{vmatrix}} t$$

and the Clifford point for $2p$ lines, $p \geqq 2$, is

$$x = \frac{\begin{vmatrix} a_1 & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-1} \end{vmatrix}}{\begin{vmatrix} a_3 & \cdots & a_{p+1} \\ \cdot & \cdot & \cdot \\ a_{p+1} & \cdots & a_{2p-1} \end{vmatrix}}.$$

The expression for the circle fails to represent a circle when the numerator of the fraction representing the radius vanishes while the denominator does not,* and in all cases where the denominator vanishes; and the expression for the Clifford point similarly fails whenever the denominator vanishes. Sets of lines for which these expressions thus fail will be called improper, all other sets proper. A necessary and sufficient condition that a set of lines be improper is given in theorem 13, § 5. The facts for 3, 4, 5, and 6 lines are well known; but a brief consideration of these cases is necessary to indicate the method of treatment of the general case and to establish a basis for an inductive proof.

For 3 lines the expression represents a circle except when $a_2 = 0$; and this is the necessary and sufficient condition that the three lines be concurrent. They meet at the point a_1.

For 4 lines the Clifford point is

$$x_1 = \frac{\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}}{a_3},$$

and this expression becomes meaningless when $a_3 = 0$. Since for 4 lines

* This case where the radius is zero is included among the cases of failure, first, because the Clifford circle is only defined by the Clifford points which lie on it, and if these points coincide they do not define a unique circle; and secondly, because of the close relationship of this case to the other cases of failure.

a_2 and a_3 are complementary, we have also $a_2 = 0$; so that both the numerator and denominator of the fraction vanish. This occurs when, and only when, the 4 lines are concurrent, meeting at the point a_1.

For 5 lines the Clifford circle is

$$x = \frac{\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}}{a_3} - \frac{\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}}{a_3} t$$

and failure occurs when

$$(1)\quad \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = 0, \qquad a_3 \neq 0;$$

$$(2)\quad a_3 = 0, \qquad \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} \neq 0, \qquad \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} \neq 0;$$

$$(3)\quad \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_3 = 0.$$

Case (1), $\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = 0$, $a_3 \neq 0$. Since $\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = 0$, we must have

$$\begin{vmatrix} c_1t_1^2 & c_1t_1 & t_1^2 & t_1 & 1 \\ c_2t_2^2 & c_2t_2 & t_2^2 & t_2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_5t_5^2 & c_5t_5 & t_5^2 & t_5 & 1 \end{vmatrix} = 0.$$

Not all of the minors of the elements in the first column can vanish. For these minors are the α_3's for the sets of 4 out of the 5 lines; and if they all vanished we would have $a_3 - a_4t = 0$ for five different values of t, and hence it would follow that $a_3 = 0$, contrary to hypothesis. Suppose the non-vanishing minor to be in the upper right-hand corner. Then the reflex point c and clinant t of each of the 5 lines must satisfy the equation

$$(\mathrm{I})\quad A_1ct^2 + A_2ct + B_1t^2 + B_2t + B_3 = 0,$$

where the A's and B's are the co-factors of the elements of the lower row, and $A_1 \neq 0$. Solving for ct we have

$$ct = (-B_1t^2 - B_2t - B_3)/(A_1t + A_2);$$

and multiplying both numerator and denominator by $1/t_1t_2t_3t_4$, this takes the form

$$(\mathrm{II})\quad ct = (\alpha t^2 + rt + \overline{\alpha})/(\gamma t + \overline{\gamma}),$$

where r is real and $\gamma \neq 0$. Since the c and t for each of the 5 lines satisfy (I), they must also satisfy (II) unless the value of t is such as to cause the denominator $\gamma t + \overline{\gamma}$ to vanish. If this is true for any t, this t must also

cause the numerator to vanish; and in this case the denominator would be a factor of the numerator. If then in equation (II) the denominator is not a factor of the numerator, this equation is satisfied by all of the 5 lines; and the 5 lines are evidently tangents of the parabola whose envelope equation is

$$tx + y = (\alpha t^2 + rt + \overline{\alpha})/z\gamma t + \overline{\gamma}).$$

If, on the other hand, $\gamma t + \overline{\gamma}$ is factor of the numerator, we may write

$$\text{(III)}\quad ct = \alpha' t + \overline{\alpha}',$$

and this equation must be satisfied by each of the 5 lines except possibly one whose clinant makes $\gamma t + \overline{\gamma}$ vanish. But all lines satisfying (III) are concurrent, and not all of our 5 lines can be concurrent since $a_3 \neq 0$. Hence 4, and only 4, of the 5 lines are concurrent.

If, then, $\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = 0$ and $a_3 \neq 0$, either the 5 lines touch a parabola or just 4 of them are concurrent; and, conversely, if 5 lines touch a parabola or just 4 of them are concurrent, then $\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = 0$ and $a_3 \neq 0$.

Case (2), $a_3 = 0$, $\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} \neq 0$, $\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} \neq 0$. Obviously $a_2 \neq 0$ and $a_4 \neq 0$; and $a_3 = 0$ gives

$$\begin{vmatrix} c_1 t_1^2 & t_1^3 & t_1^2 & t_1 & 1 \\ c_2 t_2^2 & t_2^3 & t_2^2 & t_2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_5 t_5^2 & t_5^3 & t_5^2 & t_5 & 1 \end{vmatrix} = 0.$$

Hence each of the 5 lines satisfies the equation

$$Act^2 + B_1 t^3 + B_2 t^2 + B_3 t + B_4 = 0,$$

where the A and the B's are co-factors of the elements of the lower row, and $A \neq 0$. Dividing through by $1/(t_1 t_2 t_3 t_4)^{3/2}$, and solving for ct, we have

$$ct = (\alpha_1 t^3 + \alpha_2 t^2 + \overline{\alpha}_2 t + \overline{\alpha}_1)/(rt),$$

where r is real and $\neq 0$. Since $t \neq 0$ for any line, all of the 5 lines must satisfy this equation. Moreover, we can not have $\alpha_1 = 0$; for this would give $ct = \alpha' t + \overline{\alpha}'$, which would make the 5 lines concurrent, contrary to the condition $a_2 \neq 0$. Hence all of the 5 lines must touch the deltoid whose envelope equation is

$$tx + y = (\alpha_1 t^3 + \alpha_2 t^2 + \overline{\alpha}_2 t + \overline{\alpha}_1)/(rt).$$

Conversely, if 5 lines touch a deltoid we must have $a_3 = 0$ and

$$\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} \neq 0, \qquad \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} \neq 0.$$

Case (3), $\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_3 = 0$. It follows at once that $a_2 = a_4 = 0$, and hence that the 5 lines are concurrent at the point a_1.

For 6 lines the Clifford point is

$$x_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \Big/ \begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix}$$

and failure occurs when

$$(1) \quad \begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix} = 0, \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_4 & a_5 & a \end{vmatrix} \neq 0;$$

$$(2) \quad \begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = 0.$$

Case (1), $\begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix} = 0, \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \neq 0.$ With $\begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix} = 0$ we have the complementary condition $\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = 0$. Then unless $a_3 = a_4 = 0$, all the determinants of the 2nd order in the matrix $\left\| \begin{matrix} a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{matrix} \right\|$ would vanish, causing $\begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$ to vanish. It follows that $a_3 = a_4 = 0$, and therefore also that $a_2 \neq 0$, $a_5 \neq 0$. Since $a_3 - a_4 t = 0$ for all values of t, we have $\alpha_3 = 0$ for every set of 5 put of the 6 lines. Moreover, $\begin{vmatrix} \alpha_2 & \alpha_3 \\ \alpha_3 & \alpha_4 \end{vmatrix}$ can not vanish for any set of 5; for if it did, it would follow for that set of 5 lines that $\alpha_2 = \alpha_3 = \alpha_4 = 0$ or, in terms of the a's for the 6 lines, $a_2 - a_3 t = a_3 - a_4 t = a_4 - a_5 t = 0$, contrary to the condition, $\begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \neq 0$. Hence each set of 5 out of the 6 lines touch a deltoid; and since a deltoid is uniquely determined by four of its tangents, it follows that the 6 lines all touch one deltoid. And conversely, if 6 lines touch a deltoid, we must have $a_3 = a_4 = 0$, and $a_2 \neq 0$, $a_5 \neq 0$.

Case (2), $\begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = 0$. As in case (1), $\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = 0$

also. If $a_3 = 0$, it follows that $a_4 = a_2 = a_5 = 0$, and hence that the 6 lines are concurrent. If on the contrary $a_3 \neq 0$, then $a_4 \neq 0$, $a_2 \neq 0$, $a_5 \neq 0$, and all the determinants of the 2nd order in the matrix $\left\|\begin{matrix} a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{matrix}\right\|$ vanish. Since then $\begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \\ t^2 & t & 1 \end{vmatrix} = 0$ for all values of t, we have $\begin{vmatrix} \alpha_2 & \alpha_3 \\ \alpha_3 & \alpha_4 \end{vmatrix} = 0$ for every set of 5 out of the 6 lines. Also α_3 can not vanish for more than one set of 5. There are then two possibilities in this case, viz., either all of the 6 lines touch a parabola, or just 5 of them are concurrent. And conversely, if 6 lines touch a deltoid, or if just 5 of them are concurrent, then all the determinants of 2nd order in the matrix $\left\|\begin{matrix} a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{matrix}\right\|$ vanish, and $a_2 \neq 0$, $a_3 \neq 0$, $a_4 \neq 0$, $a_5 \neq 0$.

§ 3. *The Curves* $P_k^{(m)}$.

A polynomial of the form

$$\alpha_1 t^n + \alpha_2 t^{n-1} + \cdots + \bar{\alpha}_2 t + \bar{\alpha}_1 \equiv P(t),$$

in which the coefficients of t^i and t^{n-i} are conjugates, may be called a *self-conjugate polynomial of degree n in t.* If n is even, the coefficient of $t^{n/2}$ is real. Such a polynomial has a theory analogous to that of a polynomial with real coefficients. Expressing it as a product of linear factors,

$$P(t) \equiv \alpha_1(t - \rho_1)(t - \rho_2) \cdots (t - \rho_n),$$

all of the ρ's which are not turns must pair off into inverse pairs, *i.e.*, into pairs ρ_1 and ρ_2 such that $\rho_1\bar{\rho}_2 = \bar{\rho}_1\rho_2 = 1$. Thus the number of non-turn roots of $P(t) = 0$ is always even; and if n is odd, at least one root must be a turn. Any such self-conjugate polynomial may be decomposed into self-conjugate polynomial factors, no factor being of degree higher than the second. If a self-conjugate polynomial is divisible by another self-conjugate polynomial, the quotient is also self-conjugate; and the highest common factor of two self-conjugate polynomials is a self-conjugate polynomial. It will be convenient to consider

$$\alpha_{i+1} t^{n-i} + \alpha_{i+2} t^{n-i-1} + \cdots + \bar{\alpha}_{i+2} t^{i+1} + \bar{\alpha}_{i+1} t^i$$

as a self-conjugate polynomial *of degree n.* It may be understood that 0 and ∞ are i-fold inverse roots in this case. Thus $t^3 + t^2$ is a self-conjugate polynomial of degree 5 rather than of degree 3. In breaking this polynomial up into self-conjugate factors, the factor t is to be regarded as a self-conjugate polynomial of degree 2.

The notation $P_k^{(m)}$ will indicate the curve whose envelope equation is

$$tx + y = \frac{\alpha_1 t^m + \alpha_2 t^{m-1} + \cdots + \bar{\alpha}_2 t + \bar{\alpha}_1}{\gamma_{k+1} t^{m-k-1} + \gamma_{k+2} t^{m-k} + \cdots + \bar{\gamma}_{k+2} t^{k+1} + \gamma_{k+1} t^k} \equiv \frac{\alpha(t)}{\gamma(t)}$$

where $m \geqq 1$; $0 \leqq k \leqq (m-1)/2$; $\gamma_{k+1} \neq 0$; and $\alpha(t)$ and $\gamma(t)$ are self-conjugate polynomials of degree m and $m-1$ respectively, with no common polynomial factor. From this last restriction it follows that for $k > 0$, $\alpha_1 \neq 0$; and it may also be understood to imply that $\alpha(t)$ can not vanish identically except in the rather trivial case $m = 1$, $k = 0$. The map equation of this curve is

$$x = \frac{d}{dt}\left[\frac{\alpha(t)}{\gamma(t)}\right] = \frac{\gamma(t)\alpha'(t) - \alpha(t)\gamma'(t)}{\gamma^2(t)}.$$

It is a rational curve of class m. Assuming for the moment the projective point of view, we may say that the curve is tangent to the line at infinity in the $m-1$ directions given by the roots of $\gamma(t) = 0$ (real directions for roots that are turns, pairs of imaginary directions for the pairs of inverse roots). If μ is the multiplicity of a root of $\gamma(t) = 0$, the curve has contact of the μth order in the corresponding direction. In particular, corresponding to the roots 0 and ∞, there is contact of the kth order at the circular points I and J. If σ represents the number of *distinct* roots of $\gamma(t) = 0$, the curve is of order $m + \sigma - 1$; the line at infinity counts as

$$\tfrac{1}{2}[(m-1)(m-4) + 2\sigma]$$

double tangents and as $m - \sigma - 1$ flex tangents, thus accounting for all the line singularities. There are $m + 2\sigma - 4$ cusps and

$$\tfrac{1}{2}[(m+\sigma)^2 - 7m - 9\sigma + 14]$$

nodes, these cusps and nodes not being necessarily distinct. Two such curves with the same values for m and k have m^2 common tangents, of which the line at infinity counts for at least* $(m-1)^2 + 2k$. Hence they have, aside from the line at infinity, not more than $2m - 2k - 1$ common tangents; and therefore such a curve is uniquely determined by $2m - 2k$ of its tangents. Two curves $P_k^{(m_1)}$ and $P_k^{(m_2)}$ have $m_1 m_2$ common tangents, of which the line at infinity counts for at least $(m_1 - 1)(m_2 - 1) + 2k$. Hence they have, aside from the line at infinity, not more than $m_1 + m_2 - 2k - 1$ common tangents.

This curve $P_k^{(m)}$ is the most general type of curve that can be represented by an envelope equation. It is the dual of the Jonquieres type,

* The line at infinity counts for more than $(m-1)^2 + 2k$ of the common tangents in case the equations $\gamma(t) = 0$ for the two curves have common roots other than 0 and ∞.

since it is rational and the line at infinity furnishes all of its line singularities. For $k = 0$ it is the $(m-1)$-fold parabola* of Clifford's paper. As special cases, it may be noted that a $P_0^{(1)}$ is merely a point, a $P_0^{(2)}$ is an ordinary quadratic parabola, and a $P_1^{(3)}$ is a deltoid. The curve $\Delta^{(2n-1)}$ of Morley† and Stephens‡ is a $P_{n-1}^{(2n-1)}$; and the curve $K^{(2n)}$ of Atchison§ is a $P_{n-1}^{(2n)}$.

§ 4. *Characteristic Matrices for N Lines.*

Let the positive integer p be defined by the relations $n = 2p-1$ when n is odd and $n = 2p$ when n is even; and let $[h, i]$, where $h \geqq 1$, $i \leqq p$, $i - h \geqq 0$, represent the matrix||

$$\begin{Vmatrix} a_h & a_{h+1} & \cdots & a_{n-i+1} \\ a_{h+1} & a_{h+2} & \cdots & a_{n-i+2} \\ \cdot & \cdot & \cdot & \cdot \\ a_i & a_{i+1} & \cdots & a_{n-h+1} \end{Vmatrix}.$$

For a set of n lines there is a triangular table of such characteristic matrices, as follows:

$$\begin{array}{lllll} [1,1] & & & & \\ [1,2] & [2,2] & & & \\ \vdots & \vdots & & & \\ [1,i] & [2,i] & \cdots [h,i] & \cdots & [i,i] \\ \vdots & \vdots & \vdots & & \\ [1,p-1] & [2,p-1] & \cdots [h,p-1] & \cdots & [p-1,p-1] \\ [1,p] & [2,p] & \cdots [h,p] & \cdots [p-1,p] & [p,p]. \end{array}$$

In this table, a constant h gives a column, a constant i gives a row, a constant difference $i - h$ gives a principal diagonal (running downwards to the right), and a constant sum $i + h$ gives a skew diagonal (upwards to the right). For $n = 2p-1$ the difference between the number of columns and the number of rows in each matrix is even, and the matrices of the lower row are square; while for $n = 2p$ this difference is odd, and there are no square matrices. All the matrices are self-complementary in the sense that two elements symmetrically situated with respect to the center are complementary.

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* Clifford does not state explicitly that his multi-fold parabola must not be tangent to the line at infinity at the circular points, but he ignores the consequences which would result from such specialization of his curves.

† "Orthocentric Properties of the Plane N-Line," *Trans. of the Amer. Math. Soc.*, Vol. 4, page 1; 1903.

‡ "On the Pentadeltoid," *Trans. of the Amer. Math. Soc.*, Vol. 7, page 207; 1906.

§ "Curves with a Directrix," Johns Hopkins dissertation, 1908.

|| This matrix is also a function of n, but it is simpler to omit the letter n from the symbol.

We shall be concerned with the vanishing of the determinants of highest order in these matrices, and the following notation will be used:

$[h, i] = 0$ means that every determinant of highest order vanishes;

$[h, i] \neq 0$ means that at least one determinant of highest order does not vanish;

$\{[h, i]\} = 0$ means that every solid* determinant of highest order vanishes;

$\{[h, i]\} \# 0$ means that no solid determinant of highest order vanishes;

$\}[h, i]\{ = 0$ means that at least one solid determinant of highest order vanishes;

$\{[h, i]\} \doteq 0$ means that all the solid determinants of the matrix vanish except the first and the last which do not vanish.

The two following well-known theorems will be used:

THEOREM A.† If D is a determinant of order m, A_{11}, A_{1m}, A_{m1}, and A_{mm} are the first minors of the four corner elements, and K is the second minor obtained by striking off a border one element wide all around; then

$$DK = \begin{vmatrix} A_{mm} & A_{m1} \\ A_{1m} & A_{11} \end{vmatrix}.$$

THEOREM B.‡ If in a matrix a determinant of order r does not vanish, and if every determinant of order $r + k$ which contains this non-vanishing determinant vanishes; then every determinant of order $r + k$ in the matrix vanishes.

THEOREM 1. If $[h, i] = 0$, then also $[h', i'] = 0$ for any h' and i' such that $i' + h' \geqq i + h$ and $i' - h' \geqq i - h$.

These matrices $[h', i']$ cover an area in the table of matrices which is an isosceles triangle with vertex at the matrix $[h, i]$ and base in the lowest row, or such portion of such a triangle as is not cut off by the left-hand boundary of the table. The truth of the theorem is obvious from the fact that the matrix $[h, i]$ furnishes a band extending horizontally across the matrix $[h', i']$.

THEOREM 2. If $[h, i] = 0$, then $\{[h', i']\} = 0$ for any h' and i' such that $i - h \leqq i' - h' \leqq n - h - i + 1$ and $i' \geqq i$.

The matrices $[h', i']$ of this theorem make up a parallelogram with the matrix $[h, i]$ at one vertex and its base in the lowest row, or such portion of such a parallelogram as is not cut off by the left hand boundary of the table. (For $h = 1$ this theorem adds no information to that given by theorem 1.) The proof consists in the fact that the matrix $[h, i]$ furnishes

* A solid determinant of highest order means one made up of consecutive columns of the matrix.

† Bôcher, "Higher Algebra," page 33.

‡ This theorem is a slight extension of the one given by Bôcher, "Higher Algebra," page 54.

a band extending horizontally across any solid determinant of the matrix $[h', i']$.

THEOREM 3. If $\{[h, i]\} = 0$ and $i - h \leqq 1$, then either $\{[h+1, i]\} = 0$ or $\{[h+1, i]\} \neq 0$.

The condition $i - h \leqq 1$ is needed in order that the matrix $[h+1, i]$ should exist. For $i = p$, $n = 2p - 1$, the theorem is trivial. For $i = p$, $n = 2p$, it follows from the fact that the two solid determinants in the matrix $[h+1, i]$ are complementary and must vanish together. For $i < p$, Theorem A shows that any three successive solid determinants of the matrix $[h, i-1]$ are such that the square of the middle one is equal to the product of the two adjacent ones. It follows at once that either $\{[h+1, i]\} = 0$ or $\{[h+1, i]\} \neq 0$.

The notation $\{[h, i]\} \neq 0*$ will be used to indicate that either $\{[h, i]\} \neq 0$ or else $h > i$ and therefore the matrix $[h, i]$ does not exist.

THEOREM 4. If $\{[h, i]\} = 0$ and $\{[h+1, i]\} \neq 0*$, then $\{[h, i-1]\} \neq 0*$ and $[h, i] = 0$.

If $h = i$, the matrix $[h+1, i]$ does not exist, and also the matrix $[h, i-1]$ does not exist. Also, in this case, there will be only one row in the matrix $[h, i]$, and hence the statements $\{[h, i]\} = 0$ and $[h, i] = 0$ are identical.

If $i > h$, then $\{[h+1, i]\} \neq 0$, and from the argument for the preceding theorem it follows that $\{[h, i-1]\} \neq 0$. Then the statement $[h, i] = 0$ will follow from $\{[h, i]\} = 0$ and $\{[h+1, i]\} \neq 0$ by a repeated use of Theorem B.

THEOREM 5. If $[h, i] = 0$ and $\{[h+1, i]\} \neq 0*$, and

$$\}[h + i + j - n - 2, j]\{ = 0$$

where $j \leqq i - 1$; then $[h-1, i-1] = 0$.

From $[h, i] = 0$ it follows that $\{[h', i']\} = 0$ over a parallelogram area as given in Theorem 2. The matrix $[h + i + j - n - 2, j]$ is in the principal diagonal immediately to the left of this parallelogram. The existence of this matrix gives us the additional inequality $h + i + j - n - 2 \geqq 1$. All the solid determinants of $[h-1, i-1]$ except possibly the first and last (which are complementary), vanish because of the condition $[h, i] = 0$. To prove therefore that $\{[h-1, i-1]\} = 0$ it is only necessary to show that either the first or last solid determinant in the matrix vanishes.

Let

$$\begin{vmatrix} a_{h+i+j-n-2+\lambda} & \cdots & a_{j+\lambda} \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\ a_{j+\lambda} & \cdots & a_{j+n-h-i+2+\lambda} \end{vmatrix}$$

where $0 \leqq \lambda \leqq n - 2j + 1$, be the vanishing solid determinant of the

matrix $[h+i+j-n-2,\ j]$; and consider first the case where $i=h$ and therefore $[h+1,\ i]$ does not exist. The condition $[h,\ i]=0$ in this case is simply $a_h = a_{h+1} = \cdots = a_{n-h+1} = 0$; and the vanishing determinant takes the form

$$\begin{vmatrix} a_{h+i+j-n-2+\lambda} & \cdots & a_{l-1} & 0 & \cdots & 0 \\ a_{h+i+j-n-1+\lambda} & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{h-1} & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & a_{n-h+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & a_{n-h+2} & \cdots & a_{j+n-h-i+2+\lambda} \end{vmatrix}$$

or $a_{h-1}^{r} a_{n-h+2}^{s} = 0$, where $r = n-i-j+1-\lambda$ and $s = j-i+2+\lambda$. Since a_{h-1} and a_{n-h+2} are complementary, it follows that they both vanish, and that therefore $\{[h-1,\ i-1]\} = 0$; and this is the same, in this case, as $[h-1,\ i-1]=0$.

Suppose next that $i-h>0$, and that therefore $[h+1,\ i]$ exists. The vanishing solid determinant of $[h+i+j-n-2,\ j]$ may be written in the form

$$\begin{vmatrix} a_{h+i+j-n-2+\lambda} & \cdots & a_{h-1} & a_h & \cdots & \cdots & \cdots & a_{j+\lambda} \\ a_{h+i+j-n-1+\lambda} & \cdots & a_h & a_{h+1} & \cdots & \cdots & \cdots & a_{j+\lambda+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{h-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{n-i+1} \\ a_h & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{n-i+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{j-i+h+\lambda} & \cdots & \cdots & \cdots & \cdots & a_{n-i+1} & \cdots & \cdots \\ a_{j-i+h+\lambda+1} & \cdots & \cdots & a_{n-2i+h+3} & \cdots & a_{n-i+2} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{j+\lambda} & \cdots & \cdots & a_{n-i+2} & \cdots & a_{n-h+1} & \cdots & a_{j+n-h-i+2+\lambda} \end{vmatrix}$$

From the fact that $[h,\ i]$ is of rank $i-h$, it follows that there is a (homogeneously) unique set of constants $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_{i-h+1}$ such that if the elements of each row of this matrix $[h,\ i]$ be multiplied by the corresponding λ, the sum of each column will then be zero; and one may use for the λ's the $i-h+1$ determinants of order $i-h$ in the first $i-h$ columns. If then in the above determinant we multiply the first $i-h+1$ top rows respectively by these λ's, and add for a new top row; and do the same for each successive block of $i-h+1$ rows down to the lowest block of $i-h+1$ rows; we obtain the following vanishing determinant

$$\begin{vmatrix} \cdots & \cdots & D_1 & 0 & \cdots & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ D_1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \cdots & D_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & D_2 & \cdots & \cdots \\ a_{j-i+h+\lambda+1} & \cdots & \cdots & a_{n-2i+h+3} & \cdots & a_{n-i+2} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{j+\lambda} & \cdots & \cdots & a_{n-i+2} & \cdots & a_{n-h+1} & \cdots & \cdots & a_{j+n-h-i+2+\lambda} \end{vmatrix} = 0;$$

or $D_1^{\prime} D_2^{\prime} D_3 = 0$, where

$$D_1 = \begin{vmatrix} h-1 & \cdots & i-1 \\ \cdots & \cdots & \cdots \\ i-1 & \cdots & 2i-h-1 \end{vmatrix},$$

$$D_2 = \begin{vmatrix} n-i+2 & h & h+1 & \cdots & i-1 \\ \cdots & \cdots & \cdots & & \cdots \\ n-h+2 & i & i+1 & \cdots & 2i-h-1 \end{vmatrix},$$

and

$$D_3 = \begin{vmatrix} n-2i+h+3 & \cdots & n-i+2 \\ \cdots & & \cdots \\ n-i+2 & \cdots & n-h+1 \end{vmatrix}.$$

$D_3 \neq 0$; for it is the last solid determinant in $[h, i-1]$, and from Theorem 4, $\{[h, i-1]\} \neq 0$. Hence either D_1 or D_2 must vanish. D_1 is the first solid determinant in $[h-1, i-1]$; and if $D_2 = 0$, by the repeated use of Theorem B, with the hypothesis $[h, i] = 0$, it will follow that the last solid determinant in $[h-1, i-1]$ vanishes. This completes the proof that $\{[h-1, i-1]\} = 0$. But with $\{[h, i-1]\} \neq 0$, Theorem 4 gives $[h-1, i-1] = 0$.

THEOREM 6. If $[h, i] = 0$ and $\{[h+1, i]\} \neq 0$ $*$ and

$$[h+i+i'-1, i'] = 0,$$

where $i' \geqq i$; then $[h-1, i-1] = 0$.

The matrix $[h+i-i'-1, i']$ is any matrix in the skew diagonal immediately to the left of the matrix $[h, i]$. By Theorem 2,

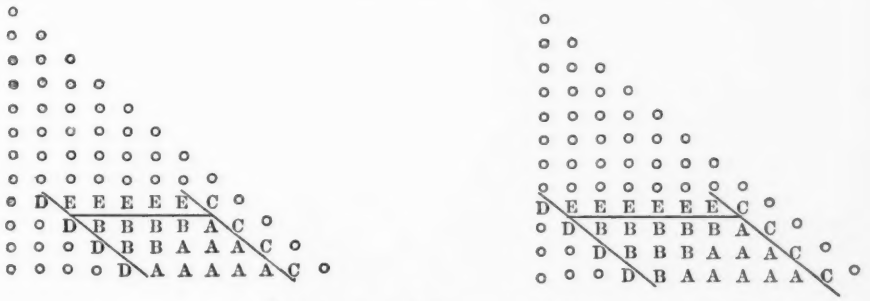
$$[h+i-i'-1, i'] = 0$$

gives $\{[h+i+i'-n-2, i']\} = 0$. This, with $[h, i] = 0$ and $\{[h+1, i]\} \neq 0$ $*$, satisfies the hypothesis of Theorem 5, and hence gives $[h-1, i-1] = 0$.

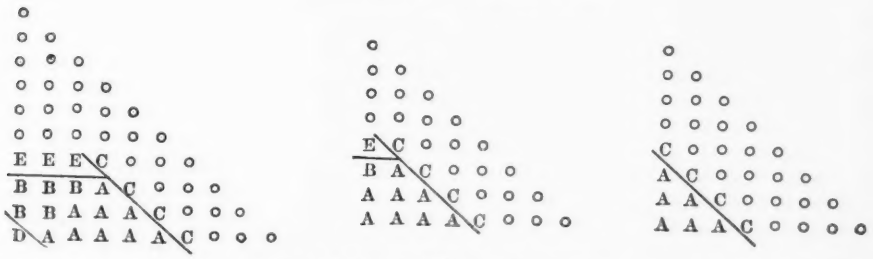
By means of the six foregoing theorems, it may be shown that any characteristic matrix for n lines which has the property $[h, i] = 0$ or

$\{[h, i]\} = 0$ lies in a "vanishing area" in the triangular table of matrices; this vanishing area being a parallelogram or portion of a parallelogram. Every such vanishing parallelogram extends downwards to include the lowest (or pth) row; and is made up of matrices of two kinds A and B, as shown in the figure below, having the respective properties $[A] = 0$ and $\{[B]\} = 0$. For n odd, the matrices in the upper right and lower left corners of the parallelogram are both A's, and lie on the same skew diagonal. For n even, the upper right matrix is an A while the lower left is a B, and the skew diagonal containing the former is immediately to the right of that containing the latter. This vanishing parallelogram is bordered on three sides by a non-vanishing border of matrices C, D, and E, having the properties respectively $\{[C]\} \neq 0 *$, $\{[D]\} \neq 0$, $\{[E]\} \doteq 0$.

A vanishing area may be only part of such a parallelogram, a part at the left being cut off by the left-hand boundary of the table, as shown in the following figures:



Vanishing parallelogram for n odd.



Vanishing parallelogram for n even.

Because of the non-vanishing border, two vanishing areas can obviously not overlap.

The matrix $[h, i]$, in which $h \geqq 2$, will be called a *key matrix*

(*a*) when $h = 2$, if $[h, i] = 0$ and $\{[h + 1, i]\} \neq 0 *$; and

(*b*) when $h > 2$, if $[h, i] = 0$, $\{[h + 1, i]\} \neq 0 *$, and $[h - 1, i] \neq 0$.

From this definition it may be seen that there is one and only one key

matrix in every vanishing area.* It is the single matrix A in the top row of the vanishing area, provided that this matrix is not in the first column. When the matrix A in the top row of a vanishing area is in the first column, the key matrix is then the matrix A in the row next below and the second column. If $[2, i]$ is a key matrix, we may or may not have $[1, \quad i] = 0$.

THEOREM 7. If $[h, i] = 0$ for a set of $n - 1$ out of n lines, then $[h, i + 1] = 0$ for the set of n lines.

In the matrix

$$\begin{Vmatrix} \alpha_h & \cdots & \alpha_{n-i} \\ \cdot & \cdot & \cdot \\ \alpha_i & \cdots & \alpha_{n-h} \end{Vmatrix}$$

for the $n - 1$ lines, all the determinants of highest order vanish. Hence in the matrix

$$\begin{Vmatrix} a_h & \cdots & a_{n-i} & t^{i-h+1} \\ \cdot & \cdot & \cdot & \cdot \\ a_i & \cdots & a_{n-h} & t \\ a_{i+1} & \cdots & a_{n-h+1} & 1 \end{Vmatrix}$$

where the a's are for the n lines, all the determinants of highest order which contain the last column vanish. It follows, by Theorem B, that all the determinants of highest order in the matrix vanish, and hence that $[h, i + 1] = 0$ for the n lines.

THEOREM 8. If $[h, i] = 0$ for n lines, and $i \equiv\!\!< n - p$; then $\{[h, i]\} = 0$ for any set of $n - 1$ out of the n lines.

The condition $i \equiv\!\!< n - p$ is needed to insure the existence of the matrix $[h, i]$ for $n - 1$ lines; for $i > n - p$ can only hold when n is odd and $i = p$, in which case the matrix $[h, i]$ would be in the lowest row of the table for n lines, and would not exist for $n - 1$ lines.

In the matrix

$$\begin{Vmatrix} a_h & \cdots & a_{n-i+1} \\ \cdot & \cdot & \cdot \\ a_i & \cdots & a_{n-h+1} \end{Vmatrix}$$

all the determinants of highest order vanish. Hence all the determinants of highest order in the matrix

$$\begin{Vmatrix} a_h & & \cdots & \cdots & a_{n-i+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_i & & \cdots & \cdots & a_{n-h+1} \\ t^{n-h-i+1} & & \cdots & t & 1 \end{Vmatrix}$$

vanish for all values of t. But when t is the clinant of one of the n lines,

* A trivial exception which will not concern us is the vanishing area in which the only matrix A is the matrix $[1, p]$.

the solid determinants of this matrix are (except for a factor which is a power of t) the solid determinants of the matrix

$$\begin{Vmatrix} \alpha_h & \cdots & \alpha_{n-i} \\ \cdot & \cdot & \cdot \\ \alpha_i & \cdots & \alpha_{n-h} \end{Vmatrix}$$

where the α's are for the remaining $n-1$ lines. Hence $\{[h, i]\} = 0$ for any set of $n-1$ out of the n lines.

THEOREM 9. If $[h, i]$ is a key matrix for n lines, where $i \leqq n-p$, then $[h, i]$ is also a key matrix for at least $n+h-i$ of the sets of $n-1$ out of the n lines.

By the preceding theorem, $\{[h, i]\} = 0$ for any set of $n-1$ out of the n lines. Hence for one of these sets of $n-1$ lines (see Theorem 3). either $\{[h+1, i]\} = 0$, or $\{[h+1, i]\} \neq 0$, or $h = i$ and $[h+1, i]$ does not exist. Suppose that $\{[h+1, i]\} = 0$ for a certain set of $n-1$ lines, Then

$$\begin{vmatrix} \alpha_{h+1} & \cdots & \alpha_i \\ \cdot & \cdot & \cdot \\ \alpha_i & \cdots & \alpha_{2i-h+1} \end{vmatrix} = 0$$

for this set, or

$$\begin{vmatrix} a_{h+1} & \cdots & \cdot & a_{i+1} \\ \cdot & \cdot & \cdot & \cdot \\ a_i & \cdots & \cdot & a_{2i-h} \\ t^{i-h} & \cdots & t & 1 \end{vmatrix} = 0$$

where the a's are for the n lines and t is the clinant of the omitted line. If this last equation held for more than $i-h$ distinct values of t, it would hold for all values of t, and we would have

$$\begin{vmatrix} a_{h+1} & \cdots & a_i \\ \cdot & \cdot & \cdot \\ a_i & \cdots & a_{2i-h-1} \end{vmatrix} = 0.$$

But this is the first solid determinant in the matrix $[h+1, i]$ for n lines, and by hypothesis $\{[h+1, i]\} \neq 0$. Hence we can not have $\{[h+1, i]\} = 0$ for more than $i-h$ sets of $n-1$ lines, and therefore we must have $\{[h+1, i]\} \neq 0*$ for at least $n+h-i$ sets. But if for a set of $n-1$ lines $\{[h, i]\} = 0$ and $\{[h+1, i]\} \neq 0*$, then, by Theorem 4, $[h, i] = 0$. For $h = 2$ this proves our theorem. For $h > 2$, it remains to show that for a set of $n-1$ lines for which $[h, i] = 0$ and $\{[h+1, i]\} \neq 0*$, we have also $[h-1, i] \neq 0$. Suppose that $[h-1, i] = 0$. Then, by Theorem 5, $[h-1, i-1] = 0$. Then, by Theorem 7, $[h-1, i] = 0$ for the n lines, which contradicts the hypothesis that $[h, i]$ is a key matrix for the n lines. Hence for our set of $n-1$ lines, $[h-1, i] \neq 0$, and $[h, i]$ is a key matrix.

THEOREM 10. If $[h, i]$ is a key matrix for all sets of $n-1$ out of n lines, then $[h, i] = 0$ for the n lines.

In the matrix

$$\begin{Vmatrix} \alpha_h & \cdots & \alpha_{n-i} \\ \cdot & \cdot & \cdot \\ \alpha_i & \cdots & \alpha_{n-h} \end{Vmatrix}$$

for any one of the sets of $n-1$ lines, all the determinants of highest order vanish. Therefore in the matrix

$$\begin{Vmatrix} a_h & \cdots & \cdot & a_{n-i+1} \\ \cdot & \cdot & \cdot & \cdot \\ a_i & \cdots & \cdot & a_{n-h+1} \\ t^{n-h-i+1} & \cdots & t & 1 \end{Vmatrix}$$

where the a's are for the n lines and t is the clinant of any one of them, all the solid determinants vanish. Each one of these solid determinants is of the $(i-h+1)$th degree in t (except for a factor which is a power of t); and since $i-h+1 < n$, it must vanish for all values of t. Hence all the solid determinants in

$$\begin{Vmatrix} a_h & \cdots & a_{n-i+1} \\ \cdot & \cdot & \cdot \\ a_i & \cdots & a_{n-h+1} \end{Vmatrix}$$

vanish, or $\{[h, i]\} = 0$ for the n lines. Then either $\{[h+1, i]\} = 0$ or $\{[h+1, i]\} \neq 0*$. Suppose $\{[h+1, i]\} = 0$. Then $[h+1, i]$ would lie in a vanishing area having a certain key matrix; and, by the preceding theorem, some sets of $n-1$ lines would have the same key matrix; and therefore for some sets of $n-1$ lines we would have $\{[h+1, i]\} = 0$. But this is contrary to the hypothesis that $[h, i]$ is a key matrix for all sets of $n-1$ lines. Therefore $\{[h+1, i]\} \neq 0*$ for the n lines, and, by Theorem 4, it follows that $[h, i] = 0$ for the n lines.

§ 5. *Necessary and Sufficient Condition for an Improper Set.*

THEOREM 11. (*A*) If $[k+2, i]$ is a key matrix for n lines, $n \geqq 3$, $0 \leqq k \leqq p-2$, $k+2 \leqq i \leqq p$; then there exists a unique integer g in the interval $n+k-i+2 \leqq g \leqq p-2$, such that g (and no more) of the n lines touch a unique curve $P_k^{(g-n+k+i-1)}$: and conversely,

(*B*) If g (and no more) of n lines touch a curve $P_k^{(g-n+k+i-1)}$, $n \geqq 3$, $0 \leqq k \leqq p-2$, $k+2 \leqq i \leqq p$, $n+k-i+2 \leqq g \leqq n$; then $[k+2, i]$ is a key matrix for the n lines.

It has been shown in § 2 that this theorem holds for $n = 3, 4, 5, 6$; and the general theorem may be proved by induction. It will first be shown that (*A*) holds for n lines if (*A*) and (*B*) both hold for $n-1$ lines.

Consider first the case where $i \leqq n - p$, and therefore $[k + 2, i]$ exists for any set of $n - 1$ out of the n lines. For at least $n + k - i + 2$ of these sets of $n - 1$ lines, $[k + 2, i]$ is a key matrix (Theorem 9). Applying (A) to one of these sets of $n - 1$ lines,* it follows that there is an integer g_1, $n + k - i + 1 \leqq g_1 \leqq n - 1$, such that g_1 of the $n - 1$ lines touch a curve $P_k^{(g_1 - n + k + i)}$. Similarly, for another set of $n - 1$ lines, g_2 of them will touch a curve $P_k^{(g_2 - n + k + i)}$. Then at least $g_1 + g_2 - n$ lines would be tangent to both of these curves, and therefore the curves can not be distinct. For these curves, if distinct, can have at most $g_1 + g_2 + 2i - 2n - 1$ common tangents (see § 3); and from $p \leqq i$ and $i \leqq n - p$ we find $g_1 + g_2 - n > g_1 + g_2 + 2i - 2n - 1$. It follows that there is a unique curve $P_k^{(g_1 - n + k + i)}$, $n + k - i + 1 \leqq g_1 \leqq n - 1$, which is touched by exactly g_1 lines out of each set of $n - 1$ lines for which $[k + 2, i]$ is a key matrix. If this is true for every set of $n - 1$ lines out of the n lines, it follows at once that $g_1 = n - 1$, and that all of the n lines touch a curve $P_k^{(k+i-1)}$. If there are sets of lines for which $[k + 2, i]$ is not a key matrix, exactly $g_1 - 1$ or $g_1 + 1$ lines† of such a set must touch the $P_k^{(g_1 - n + k + i)}$. In the former case, (B) says that $[k + 2, i + 1]$ would be a key matrix for the $n - 1$ lines, and this would give $\{[k + 2, i]\} \neq 0$* (Theorem 4). But since $[k + 2, i]$ is a key matrix for the n lines, we have $\{[k + 2, i]\} = 0$ for all sets of $n - 1$ lines (Theorem 8). Hence if there are any sets of $n - 1$ lines for which $[k + 2, i]$ is not a key matrix, exactly $g_1 + 1$ lines of each such set must touch the $P_k^{(g_1 - n + k + i)}$. In this case it is evident that exactly $g_1 + 1$ of the n lines touch this $P_k^{(g_1 - n + k + i)}$, and hence (A) holds for the n lines with $g = g_1 + 1$.

We must next consider the case $i > n - p$, which only occurs when $n = 2p - 1$ and $i = p$. We wish then to show that if $[k + 2, p]$ is a key matrix for $2p - 1$ lines, $2p - 1 \geqq 3$, $0 \leqq k \leqq p - 2$; then exactly g of the lines touch a curve $P_k^{(g + k - p)}$, where $p + k + 1 \leqq g \leqq 2p - 1$.

We have

$$\begin{vmatrix} a_{k+2} & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-k-2} \end{vmatrix} = 0,$$

and hence

$$\begin{vmatrix} c_1 t_1^{p-1} & \cdots & c_1 t_1^{k+1} & t_1^{p+k-1} & \cdots & t_1 & 1 \\ c_2 t_2^{p-1} & \cdots & c_2 t_2^{k+1} & t_2^{p+k-1} & \cdots & t_2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{2p-1} t_{2p-1}^{p-1} & \cdots & c_{2p-1} t^{k+1} & t_{2p-1}^{p+k-1} & \cdots & t_{2p-1} & 1 \end{vmatrix} = 0.$$

* The inequality conditions of the hypothesis are readily seen to hold for the $n - 1$ lines, provided we assume, as of course we may, that $n > 3$.

† Any set of $n - 1$ lines must evidently contain either $g_1 - 1$, g_1, or $g_1 + 1$ lines touching the $P_k^{(g - n + k + i)}$; and it can not be g_1, because this, by (B), would make $[k + 2, i]$ the key matrix.

All the minors of the elements of the first column can not vanish. For they are (except for a non-vanishing factor) the determinants

$$\begin{vmatrix} \alpha_{k+3} & \cdots & \alpha_p \\ \cdot & \cdot & \cdot \\ \alpha_p & \cdots & \alpha_{2p-k-3} \end{vmatrix}$$

for the sets of $2p-2$ out of the $2p-1$ lines; and if they all vanished we would have, in terms of the constants for the $2p-1$ lines,

$$\begin{vmatrix} a_{k+3} & \cdots & a_p & a_{p+1} \\ \cdot & \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-k-3} & a_{2p-k-2} \\ t^{p-k-2} & \cdots & t & 1 \end{vmatrix} = 0$$

for $2p-1$ different values of t. This would therefore be an identity in t, and we would have

$$\begin{vmatrix} a_{k+3} & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-k-3} \end{vmatrix} = 0$$

for the $2p-1$ lines, contrary to the hypothesis. Suppose the non-vanishing minor to be in the upper right-hand corner. Then the reflex point c and clinant t of each of the $2p-1$ lines satisfies the equation

$$\text{(I)} \qquad \begin{aligned} &A_1 c t^{p-1} + A_2 c t^{p-2} + \cdots + A_{p-k-1} c t^{k+1} + B_1 t^{p+k-1} \\ &\qquad + B_2 t^{p+k-2} + \cdots + B_{p+k-1} t + B_{p+k} = 0 \end{aligned}$$

where the A's and B's are the co-factors of the elements of the lower row, and $A_1 \neq 0$. Solving for ct, we have

$$ct = \frac{-B_1 t^{p+k-1} - B_2 t^{p+k-2} - \cdots - B_{p+k}}{A_1 t^{p-2} + A_2 t^{p-3} + \cdots + A_{p-k-1} t^k}$$

and multiplying numerator and denominator of the fraction by

$$\frac{(\sqrt{-1})^{p-1}}{(t_1 t_2 t_3 \cdots t_{2p-2})^{(p+k-1)/2}},$$

they both become self-conjugate, and we have

$$\text{(II)} \qquad ct = \frac{\alpha_1 t^{p+k-1} + \alpha_2 t^{p+k-2} + \cdots + \bar{\alpha}_2 t + \bar{\alpha}_1}{\gamma_{k+1} t^{p-2} + \gamma_{k+2} t^{p-3} + \cdots + \bar{\gamma}_{k+1} t^k}$$

where $\gamma_{k+1} \neq 0$.

The reflex point c and clinant t of each of the $2p-1$ lines must satisfy (II), unless the clinant is such that it causes the denominator and numerator to vanish. If then the numerator and denominator have no common polynomial factor, all of the $2p-1$ lines satisfy (II), and hence all of

them are tangents of the curve $P_k^{(k+p-1)}$ whose envelope equation is

$$tx + y = \frac{\alpha_1 t^{p+k-1} + \cdots + \bar{\alpha}_1}{\gamma_{k+1} t^{p-2} + \cdots + \bar{\gamma}_{k+1} t^k}$$

which is in accordance with part (A) of our theorem.

Suppose now that the numerator in (II) is not identically zero, and that the numerator and denominator have a highest common factor of degree $\mu + \nu$, containing t^ν (but not $t^{\nu+1}$) as a factor. It is obvious that μ and ν must satisfy the inequalities

$$0 \leqq \mu \leqq p - k - 2, \qquad 0 \leqq \nu \leqq k.$$

We may then take out this common factor and write*

$$\text{(III)} \qquad ct = \frac{\alpha_1' t^{p+k-\mu-2\nu-1} + \cdots + \bar{\alpha}_1'}{\gamma_{k-\nu-1}' t^{p-\mu-\nu-2} + \cdots + \bar{\gamma}_{k-\nu-1}' t^{k-\nu}}.$$

Each of the $2p - 1$ lines, satisfying (I), must satisfy (III) unless its clinant is such as to cause the removed common factor to vanish. Since the clinant of a line can not be zero, it follows that at least $2p - \mu - 1$ of them satisfy (III). Say that exactly $2p - \mu + \sigma - 1$ of the lines satisfy (III), $0 \leqq \sigma \leqq \mu$. These $2p - \mu + \sigma - 1$ lines evidently touch a curve $P_{k-\nu}^{(p+k-\mu-2\nu-1)}$. Then either all of the $2p - 1$ lines touch this curve, and therefore by (B) $[k - \nu + 2,\ p - \sigma - \nu]$ is a key matrix for every set of $2p - 2$ lines; or in at least one set of $2p - 2$ lines exactly $2p - \nu + \sigma - 1$ lines touch the curve, and hence $[k - \nu + 2,\ p - \sigma - \nu - 1]$ is a key for this set of $2p - 2$. In either case it follows that $[k - \nu + 2,\ p - \sigma - \nu] = 0$ for the $2p - 1$ lines (by Theorem 10 or 7). With $[k - \nu + 2,\ p - \sigma - \nu] = 0$, $\sigma \geqq 1$ would give (Theorem 1) $[k + 2,\ p - 1] = 0$, contrary to hypothesis; and hence we must have $\sigma = 0$. With $\sigma = 0$, $\nu \geqq 1$ would give $[k + 1,\ p] = 0$; and since $\nu \geqq 1$ would necessitate $k \geqq 1$, this again contradicts the hypothesis that $[k + 2,\ p]$ is a key matrix. It follows that in equation (III) we must have $\nu = 0$, and that the corresponding envelope equation

$$tx + y = \frac{\alpha_1' t^{p+k-\mu-1} + \cdots + \bar{\alpha}_1'}{\gamma_{k-1}' t^{p-\mu-2} + \cdots + \bar{\gamma}_{k-1}' t^k}$$

represents a curve $P_k^{(p+k-\mu-1)}$ which is touched by exactly $2p - \mu - 1$ of the $2p - 1$ lines. This is in agreement with part (A) of our theorem.

There remains the case where the numerator in equation (II) vanishes identically. In this case, at least $2p - 1 - (p - k - 2)$, or $p + k + 1$, of the $2p - 1$ lines are concurrent at the origin, *i.e.*, touch a $P_0^{(1)}$. Say

* The assumption that t^ν is a factor of the numerator gives $\alpha_1 = \alpha_2 = \cdots = \alpha_\nu = 0$; and when we then divide out t^ν, the degree of the numerator is reduced by 2ν.

that exactly $p + k + \sigma + 1$ of them are concurrent. As in the preceding paragraph, a contradiction will result unless $k = \sigma = 0$. If then the numerator in (II) vanishes identically, exactly $p + 1$ of the $2p - 1$ lines must touch a curve $P_0^{(1)}$.

This completes the proof that if (A) and (B) hold for $n - 1$ lines, then (A) must hold for n lines. It will now be shown that if (A) and (B) hold for $n - 1$ lines and (A) holds for n lines, then (B) also holds for n lines.

If g out of n lines touch a curve $P_k^{(g-n+k+i-1)}$, then in g of the sets of $n - 1$ out of the n lines, $g - 1$ lines touch the curve, while in the remaining $n - g$ sets, g lines touch the curve. Hence, applying (B) to the sets of $n - 1$ lines, $[k + 2, i]$ is a key matrix for g of the sets and $[k + 2, i - 1]$ for the remaining $n - g$ sets. It follows (by Theorem 7 if $g < n$, and by Theorem 10 if $g = n$) that $[k + 2, i] = 0$ for the n lines. There must then be a key matrix $[h', i']$ for the n lines, such that $i' + h' \equiv k + i + 2$ and $i' - h' \equiv i - k - 2$. Hence, applying (A) to the n lines, and proceeding as above, we have that either $[h', i']$ is a key matrix for all of the sets of $n - 1$ lines or $[h', i' - 1]$ is a key matrix for some sets of $n - 1$ lines. In the former case there would be a set of $n - 1$ lines with both $[h', i']$ and $[k + 2, i]$ as key matrices; while in the latter case there would be a set of $n - 1$ lines with key matrices $[h', i' - 1]$ and either $[k + 2, i]$ or $[k + 2, i - 1]$. Either case is a contradiction unless $[h', i']$ is $[k + 2, i]$. It follows that if g of the lines touch a curve $P_k^{(g-n+k+i-1)}$, $[k + 2, i]$ is a key matrix. If then (A) and (B) hold for $n - 1$ lines, they both hold for n lines; and this establishes our theorem.

THEOREM 12. If $[k + 2, i]$ is a key matrix for n lines, then for g of the sets of $n - 1$ out of the n lines (where $n + k - i + 2 \equiv g \equiv n$) $[k + 2, i]$ is a key matrix, and for the remaining $n - g$ sets $[k + 2, i - 1]$ is a key matrix.

This theorem is an obvious corollary of Theorem 11.

As noted at the beginning of § 3, a set of $2p$ lines is improper when, and only when, the determinant

$$\begin{vmatrix} a_3 & \cdots & a_{p+1} \\ \cdot & \cdot \quad \cdot & \cdot \\ a_{p+1} & \cdots & a_{2p-1} \end{vmatrix}$$

vanishes. Since this determinant is one of the two complementary solid determinants in the matrix $[2, p]$, an improper set will occur when, and only when, $\{[2, p]\} = 0$. For $2p - 1$ lines to be improper, either or both of the conditions $[2, p] = 0$, $[3, p] = 0$, must hold. And since these vanishing matrices must lie in a vanishing area having a key matrix, it is evident that the necessary and sufficient condition for an improper set is

that there should be a key matrix $[h, i]$ somewhere in the area $h \geqq 2$, $i \leqq p$, $i + h \leqq p + 3$, and $i - h \geqq 0$. Since vanishing areas can not overlap, it is evident that there could not be more than one key matrix in this area. And from Theorem 11 we then have

THEOREM 13. A necessary and sufficient condition for an improper set of n lines is that there should exist a unique set of integers k, i, and g, satisfying the inequalities

$$0 \leqq k \leqq p - 2, \qquad k + i + 2 \leqq p + 3,$$

$$k + 2 \leqq i \leqq p, \qquad n + k - i + 2 \leqq g \leqq n,$$

and such that g (and no more) of the n lines are tangent to a unique curve $P_k^{(g-n+k+i-1)}$.

§ 6. *Improper Sets as Limiting Cases.*

A set of n variable lines with characteristic constants $\alpha_1, \alpha_2, \cdots, \alpha_n$ and clinants $\tau_1, \tau_2, \cdots, \tau_n$ will be said to approach a set of n fixed lines with constants $a_1, a_2, \cdots, a_n$ and clinants $t_1, t_2, \cdots, t_n$, if the α's and τ's simultaneously approach the corresponding a's and t's. Since there are only a finite number of conditions for improper sets, it is always possible (in an infinite number of ways) to find a sequence of proper sets which approach any given improper set. The question then arises as to whether, when a variable proper set thus approaches a given improper set, the Clifford figure (point or circle) of the variable set will approach, under any circumstances, a unique limit which might be conveniently defined as the Clifford figure of the improper set. This question will be answered without giving the rather tedious details of the proofs. The results may be most conveniently expressed if we assume for the time being the point of view of the geometry of inversion; *i.e.*, we assume that the plane has only one point at infinity, and understand the word circle to include straight lines and point-circles (circles of zero radius).

Type 1; $2p - 1$ lines, key matrix $[k + 2, p - k + 1]$, $k \geqq 1$. This key matrix lies on the skew diagonal $h + i = p + 3$; and g of the $2p - 1$ lines, $p + 2k \leqq g \leqq 2p - 1$, touch a curve $P_k^{(g-p+1)}$. In the fractions giving the center and radius of the Clifford circle, the denominators vanish while the numerators do not. Of the sets of $2p - 2$ out of the $2p - 1$ lines, g sets are proper sets; but the remaining $2p - g - 1$ sets are of the type 4 described below, with their Clifford points at infinity. In this case the g finite Clifford points lie on a straight line.* When a proper set of $2p - 1$ lines approaches such an improper set in any way, the Clifford circle approaches this straight line as a limit. We may define this line therefore as the Clifford figure in this case. The reflex point of this line is

* See Morley, loc. cit.

$$c = \begin{vmatrix} a_1 & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-1} \end{vmatrix} \Big/ \begin{vmatrix} a_3 & \cdots & a_{p+1} \\ \cdot & \cdot & \cdot \\ a_{p+1} & \cdots & a_{2p-1} \end{vmatrix}.$$

The denominator of this fraction is a solid determinate of the matrix $[1, p-1]$, and can not vanish, since with $[k+2, p-k+1]$ as a key matrix we have $\{[1, p-1]\} \neq 0$. As examples of sets of this type we have 5 lines touching a deltoid; 7 lines touching a $P_2^{(4)}$; 7 lines, 6 of which touch a deltoid; etc.

Type 2; $2p-1$ lines, key matrix $[k+2, p-k]$, $k \geqq 1$. The key matrix lies on the skew diagonal $h+i = p+2$; and g of the $2p-1$ lines, $p+2k+1 \leqq g \leqq 2p-1$, touch a curve $P_k^{(g-p)}$. Both numerators and denominators of the fractions in the expression for the Clifford circle vanish. Of the sets of $2p-2$ lines, g sets are of the type 4 and the rest of the type 5 described below, with their Clifford points at infinity in both cases. When a proper set of lines approaches a set of this kind in any way, the entire Clifford circle recedes to infinity. We may therefore define the Clifford figure for this type to be a point-circle at infinity. The simplest example of a set of this type is the case of 7 lines touching a deltoid.

Type 2′; $2p-1$ lines, key matrix $[2, p]$. In this type, g of the lines, $p+1 \leqq g \leqq 2p-1$, touch a curve $P_0^{(g-p)}$; and the radius of the Clifford circle is zero.* Of the sets of $2p-2$ lines, g sets are proper, and the rest are of type 5′ with a finite Clifford point as defined below. All of these $2p-1$ Clifford points for the sets of $2p-2$ lines coincide at the point

$$\begin{vmatrix} a_1 & \cdots & a_{p-1} \\ \cdot & \cdot & \cdot \\ a_{p-1} & \cdots & a_{2p-3} \end{vmatrix} \Big/ \begin{vmatrix} a_3 & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-3} \end{vmatrix},$$

which is the focus† of the curve $P_0^{(g-p)}$. When a variable proper set approaches such a set, the center of the Clifford circle approaches this point while its radius approaches zero. We may define the Clifford figure for this type to be a point-circle at this point.

Type 3; $2p-1$ lines, key matrix $[k+2, p-k-1]$, $k \geqq 1$. This key matrix lies on the skew diagonal $h+i = p+1$; and g of the lines, $p+2k+2 \leqq g \leqq 2p-1$, touch the curve $P_k^{(g-p-1)}$. Here again both the numerators and denominators of the fractions for the center and radius of the Clifford circle vanish. Of the sets of $2p-2$ lines, g sets are of type 5 with their Clifford points at infinity, and the rest are of type 6 with entirely indeterminate Clifford points. Given a set of this type, a proper set may be made to approach it in such a way that the Clifford circle will approach any arbitrarily preassigned straight line. The simplest example of this type is a set of 9 lines touching a deltoid.

* See note, beginning of § 2.

† See Clifford, loc. cit. The denominator of this fraction can not vanish.

Type 3′; $2p-1$ lines, key matrix $[2,\ p-1]$. In this case g of the lines, $p+2 \leqq g \leqq 2p-1$, touch a curve $P_0^{(g-p-1)}$; and again the Clifford circle fractions take the indeterminate form. Of the sets of $2p-2$ lines, g sets have a Clifford point as defined in type 5′, while the remaining sets are of type 6 with indeterminate Clifford points. The g Clifford points coincide at the point

$$\begin{vmatrix} a_1 & \cdots & a_{p-2} \\ \cdot & \cdot & \cdot \\ a_{p-2} & \cdots & a_{2p-5} \end{vmatrix} \Big/ \begin{vmatrix} a_3 & \cdots & a_{p-1} \\ \cdot & \cdot & \cdot \\ a_{p-1} & \cdots & a_{2p-5} \end{vmatrix},$$

which is the focus of the curve $P_0^{(g-p-1)}$. As a proper set of lines approaches a set of this type, the circumference of the Clifford circle will always approach this point. We may then say that the Clifford circle for this type is indeterminate except that it must pass through this point. For example, if a set of 7 lines touch a parabola, a proper set may be made to approach these 7 lines in such a way that the Clifford circle will approach any preassigned circle or straight line through the focus of the parabola, or a point-circle at the focus. A still simpler example is that of 5 concurrent lines.

Type 4; $2p$ lines, key matrix $[k+2,\ p-k+1]$, $k \geqq 1$. The key matrix is on the skew diagonal $h+i=p+3$; and g of the lines, $p+2k+1 \leqq g \leqq 2p$, touch a curve $P_k^{(g-p)}$. In the fraction representing the Clifford point the denominator vanishes while the numerator does not. Of the sets of $2p-1$ out of the $2p$ lines, g sets are of type 1 and have Clifford lines, while the remaining sets are of type 2 having Clifford point-circles at infinity. As a proper set approaches a set of this type, the Clifford point moves off to infinity; and hence one may define the point at infinity to be the Clifford point for this type. The simplest example is a set of 6 lines touching a deltoid.

The g Clifford lines for the g sets of $2p-1$ lines in this case exhibit a very interesting property which is treated in the next section.

Type 5; $2p$ lines, key matrix $[k+2,\ p-k]$, $k \geqq 1$. This key matrix lies in the skew diagonal $h+i=p+2$; and g of the lines, $p+2k+2 \leqq g \leqq 2p$, touch a curve $P_k^{(g-p-1)}$. The numerator and denominator of the fraction giving the Clifford point both vanish. Of the sets of $2p-1$ lines, g are of type 2 and the rest of type 3, having respectively the point at infinity and indeterminate straight lines for their Clifford circel. An iso type 4, we may define the Clifford point for this type to be the pinst at infinity. The simplest example is a set of 8 lines tangent to a deltoid.

Type 5′; $2p$ lines, key matrix $[2,\ p]$. Here g of the lines, $p+2 \leqq g \leqq 2p$, touch a curve $P_0^{(g-p-1)}$, and the expression for the Clifford point

takes the indeterminate form. Of the sets of $2p-1$ lines, g sets are of type $2'$ and the rest of type $3'$; the former having Clifford point-circles and the latter indeterminate circles passing through fixed points. These points for all of the sets of $2p-1$ lines coincide at the point

$$\begin{vmatrix} a_1 & \cdots & a_{p-1} \\ \cdot & \cdots & \cdot \\ a_{p-1} & \cdots & a_{2p-3} \end{vmatrix} \Big/ \begin{vmatrix} a_3 & \cdots & a_p \\ \cdot & \cdots & \cdot \\ a_p & \cdots & a_{2p-3} \end{vmatrix},$$

which is the focus of the curve $P_0^{(g-p-1)}$. When a proper set approaches a set of this type in any way, the Clifford point will always approach this point of coincidence; and it may therefore be defined as the Clifford point for a set of this type. As simple examples we have 4 concurrent lines; 6 lines, 5 of which are concurrent; 6 lines touching a parabola; etc.

Type 6 consists of all improper sets not included in the preceding types. The key matrix will lie somewhere above the skew diagonal $h+i=n-p+2$. In the fractions in the expressions for the Clifford figure, both numerator and denominator will vanish. Given such an improper set of lines, one may arbitrarily preassign a Clifford figure, and then cause a proper set to approach the improper set in such a way that its Clifford figure will approach the preassigned figure. The Clifford figure for this type is therefore entirely indeterminate.

To summarize, we may say that a unique Clifford figure exists for every proper set, and, as defined in this section, for every improper set of types 1, 2, $2'$, 4, 5, and $5'$; that it is partially indeterminate for sets of type 3 and $3'$; and that it is entirely indeterminate for sets of type 6. The existence in this sense of a uniquely defined Clifford figure for a set of n lines does not imply (even for a proper set) the existence of such a determinate Clifford figure for every sub-set of lines. But it does imply

(1) That a determinate Clifford figure exists for at least $p+1$ of the sets of $n-1$ lines;

(2) That these Clifford figures which exist for sets of $n-1$ lines are all incident with the Clifford figure for the n lines;

(3) That when a variable proper set of n lines approaches the given set in any way, the Clifford figure of the variable set will always approach that of the given set.

§ 7. *The Reciprocal Case.*

In type 4 of the preceding section, g of the $2p$ lines touch a curve $P_k^{(g-p)}$, and g of the sets of $2p-1$ lines (the sets obtained by omitting one by one the g lines which touch the curve) have Clifford lines instead of circles. The reflex point of one of these lines (see type 1) is

$$c = \begin{vmatrix} a_1 & \cdots & a_p & t^p \\ \cdot & \cdots & \cdot & \cdot \\ a_p & \cdots & a_{2p-1} & t \\ a_{p+1} & \cdots & a_{2p} & 1 \end{vmatrix} \Big/ \begin{vmatrix} a_3 & \cdots & a_{p+1} & t^{p-1} \\ \cdot & \cdots & \cdot & \cdot \\ a_{p+1} & \cdots & a_{2p-1} & t \\ a_{p+2} & \cdots & a_{2p} & 1 \end{vmatrix}$$

where the a's are for the $2p$ lines and t is the clinant of the omitted line. With $[k+2,\ p-k+1]$ as key matrix, the coefficients of the highest and lowest powers of t in the denominator vanish, so that we may write

$$c = \frac{A_p t^p + A_{p-1}t^{p-1} + \cdots + A_1 t + A_0}{B_{p-k}t^{p-k} + B_{p-k-1}t^{p-k-1} + \cdots + B_{k+1}t^{k+1}}$$

where, moreover, $A_p \neq 0$ and $B_{p-k} \neq 0$. The clinant τ of this line is equal to the quotient $\bar{c}/c$. Under the condition $[k+2,\ p-k+1] = 0$, it may be seen that the ratio $B_s/\bar{B}_{p-s+1}$ is a constant turn for all values of s; and if we therefore put $B_s/\bar{B}_{p-s+1} = T$, we find

$$\tau = \frac{\bar{c}}{c} = (-S)^p T t,$$

where S is the product of the $2p$ clinants of the lines of the set. We may then put $t = \tau/(-S)^p T$, and thus obtain an expression for the reflex point c of the line in terms of its clinant τ, thus

$$c = \frac{C_p t^p + C_{p-1}t^{p-1} + \cdots + C_1 t + C_0}{D_{p-k}t^{p-k} + \cdots + D_{k+1}t^{k+1}},$$

where

$$C_j = A_j/(-S)^{pj}T^j \qquad \text{and} \qquad D_s = B_s/(-S)^{ps}T^s.$$

If we now multiply both sides of the equation by τ, and the numerator and denominator of the fraction by $(-S)^{(p^2+p)/2}T^{p/2}$, we obtain

$$c\tau = \frac{\alpha_1\tau^p + \alpha_2\tau^{p-1} + \cdots + \bar{\alpha}_2\tau - \bar{\alpha}_1}{\gamma_{k+1}\tau^{p-k-1} + \cdots + \bar{\gamma}_{k+1}\tau^k},$$

where the numerator and denominator are now found to be self conjugate, and $\alpha_1 \neq 0$ and $\gamma_{k+1} \neq 0$. Hence we see that these g Clifford lines are tangents of the curve $P_k^{(p)}$ whose envelope equation is

$$tx + y = \frac{\alpha_1\tau^p + \alpha_2\tau^{p-1} + \cdots + \bar{\alpha}_1}{\gamma_{k+1}\tau^{p-k-1} + \cdots + \bar{\gamma}_{k+1}\tau^k}.$$

The most interesting case occurs when $g = 2p$, *i.e.*, when all of the $2p$ lines touch a curve $P_k^{(p)}$. We then obtain $2p$ Clifford lines for the sets of $2p-1$ lines, and these Clifford lines touch a new curve $P_k^{(p)}$. An examination of the incidence relations of the various Clifford points and circles for the subsets of lines shows that the relation between the two

sets of $2p$ lines is reciprocal; *i.e.*, if we start with the $2p$ Clifford lines, we are led back to the original set.*

There is also a certain reciprocity in the cases where $g < 2p$. For instance, when $g = 2p - 1$, $2p - 1$ of the $2p$ lines touch a $P_k^{(p-1)}$, and we are led to $2p - 1$ Clifford lines touching a $P_k^{(p)}$. Reversing the process, and starting with these $2p - 1$ lines, we have a case of type 1 with a Clifford line, this line being the one line of the original set which does not touch the $P_k^{(p-1)}$. For $g < 2p - 1$, the fact that the g Clifford lines touch a $P_k^{(p)}$ is not sufficient to make them an improper set;† and hence they will, in general, be a proper set and have a Clifford element which will be the Clifford element of the $2p - g$ lines of the original set which do not touch the $P_k^{(g-p)}$.

For example, suppose that 19 of 24 lines touch a curve $P_3^{(7)}$, while the remaining 5 lines have no exceptional conditions upon them. Then 19 of the sets of 23 lines will be of type 1 and have Clifford lines, and these 19 Clifford lines will touch a $P_3^{(12)}$. Starting with these 19 lines, they will be found to be a proper set, their Clifford circle being that of the 5 lines of the original set which did not touch the $P_3^{(7)}$. But if the 5 lines of the original set touch a parabola, and hence have a point-circle as their Clifford figure, then the 19 lines in the reverse process will lead to this same point-circle as their Clifford figure, and will therefore be of type 2′; and it follows that in addition to the fact that the 19 Clifford lines touch a curve $P_3^{(12)}$, g of them, where $11 \leqq g \leqq 19$, must also touch a curve $P_0^{(g-10)}$.

§ 8. *The Curve $P_k^{(m)}$ as a Sum of Fundamental Curves.*

Let two curves $P_{k_1}^{(m_1)}$ and $P_{k_2}^{(m_2)}$ be given by the envelope equations

$$tx + y = \frac{\alpha_1(t)}{\gamma_1(t)} \qquad \text{and} \qquad tx + y = \frac{\alpha_2(t)}{\gamma_2(t)};$$

and suppose that $\gamma_1(t)$ and $\gamma_2(t)$ have no common polynomial factor, which necessitates that one of the k's, say k_2, should be zero. Then the envelope equation

$$tx + y = \frac{\alpha_1(t)}{\gamma_1(t)} + \frac{\alpha_2(t)}{\gamma_2(t)} = \frac{\alpha_1(t)\gamma_2(t) + \alpha_2(t)\gamma_1(t)}{\gamma_1(t)\gamma_2(t)}$$

represents a curve $P_{k_1}^{(m_1+m_2-1)}$ which may be called the "sum" of the two curves. Since the map equations of the three curves $P_{k_1}^{(m_1)}$, $P_0^{(m_2)}$, and $P_{k_1}^{(m_1+m_2-1)}$ are respectively

* Kantor (loc. cit.) discovered this property for a set of 6 lines touching a deltoid. Morley generalized Kantor's theorem; but his generalization is quite different from the one here given.

† For $g \leqq 2p - 2k$, the conclusion that the g lines touch a curve $P_k^{(p)}$ is obviously trivial.

$$x = \frac{d}{dt}\left(\frac{\alpha_1(t)}{\gamma_1(t)}\right), \quad x = \frac{d}{dt}\left(\frac{\alpha_2(t)}{\gamma_2(t)}\right), \quad \text{and} \quad x = \frac{d}{dt}\left(\frac{\alpha_1(t)}{\gamma_1(t)}\right) + \frac{d}{dt}\left(\frac{\alpha_2(t)}{\gamma_2(t)}\right),$$

and since in all three cases the parameter t is the clinant of the tangent to the curve at the point x, the geometric significance of the addition is obvious. If we take two points, one on the curve $P_{k_1}^{(m_1)}$ and the other on the curve $P_0^{(m_2)}$, where the tangents to the curves are parallel, and add these points by the ordinary construction for the sum of two complex numbers; the result will be the point of the curve $P_{k_1}^{(m_1+m_2-1)}$ at which the tangent has the same direction.* We may similarly define the sum of any number of such curves, all the denominators in the envelope equations being prime to each other.

We may call a curve $P_k^{(m)}$ a "fundamental curve" when the denominator $\gamma(t)$ in the envelope equation is a power of a self-conjugate factor of the first degree or a power of a self-conjugate factor of the second degree which is not the product of self-conjugate factors of the first degree. For such a fundamental curve we have $k = 0$, with $\gamma(t)$ of the form $(\gamma t + \overline{\gamma})^{m-1}$ or $(\gamma t^2 + rt + \overline{\gamma})^{(m-1)/2}$; or† $k = \frac{1}{2}(m-1)$, with $\gamma(t)$ of the form rt^k. The map equations in the three cases may be written respectively in the forms

$$x = A_0 + \frac{B_2}{(\gamma t + \overline{\gamma})^2} + \frac{B_3}{(\gamma t + \overline{\gamma})^3} + \cdots + \frac{B_m}{(\gamma t + \overline{\gamma})^m},$$

$$x = A_0 + \frac{B_{m-1}t^{m-1} + B_{m-2}t^{m-2} + \cdots + B_1 t + B_0}{(\gamma t^2 + rt + \overline{\gamma})^{m+1/2}},$$

$$x = A_k t^k + A_{k-1}t^{k-1} + \cdots + A_1 t + A_0 + \frac{B_1}{t} + \frac{B_2}{t^2} + \cdots + \frac{B_{k+1}}{t^{k+1}}.$$

In the first two cases the point A_0 is the focus of the curve; while in the third case it may be called the center (suggested by the center of the deltoid when $k = 1$). When $A_0 = 0$, *i.e.*, when the focus or center is at the origin, the curve may be said to be in canonical position. Every curve $P_k^{(m)}$ is the sum of q such fundamental curves,‡ where q is the number of distinct irreducible self-conjugate factors of $\gamma(t)$. It is evident that if we had such a decomposition of a curve $P_k^{(m)}$ into a sum of fundamental curves, the A_0's for the several fundamental curves would be arbitrary except that their sum would be fixed. The point represented by the sum of the A_0's may be called (for lack of a better name) the center of the $P_k^{(m)}$; and the $P_k^{(m)}$ may be said to be in canonical position when its center is at the origin. If a curve $P_k^{(m)}$ is in canonical position, its decomposition into the sum of fundamental curves in canonical position is unique.

* These curves have one and only one tangent in a given direction.

† This case $k = \frac{1}{2}(m-1)$ is the curve $\Delta^{(2n-1)}$ of Morley and Stephens (loc. cit.).

‡ Theorems 5 and 6 of Atchison's paper (loc. cit.) are special cases of this.

ON THE REPRESENTATIONS OF NUMBERS AS SUMS OF 3, 5, 7, 9, 11 AND 13 SQUARES.*

By E. T. Bell.

I. Preliminary Considerations.

1. Of the nine sections into which this paper is divided, I is preliminary; II contains eleven general formulas of a simple nature; III–VIII apply the first seven formulas of II to the determination in finite form of the number of representations of an integer as a sum of 3, 5, 7, 9, 11 and 13 squares respectively, attention being paid to the numbers of odd squares in the representations; and IX is devoted to recurrences, consequences of the last four formulas of II, for facilitating the numerical computations implied in III–VIII. A complete system of results of a well-defined kind, described in §§ 4–6, is obtained in III–VIII; but in the sense of determining finitely the total number of representations for any linear form of the integer to be represented, the enumerations for 11 and 13 squares are only partial. If complete finite systems are desired in these and further odd cases, they may be found by applying II to the theorems of H. J. S. Smith, Glaisher, and others, concerning an even number of squares. The resulting formulas, however, are simple neither in the common meaning nor in the technical sense defined in § 4; and as they belong to a wholly different order of ideas, they are not included here.

From one point of view the formulas, particularly the recurrences, for 5, 7, 9, 11 and 13 squares may be looked upon as analogues of the class number relations of Kronecker,† which, as has long been known, are intimately related to the decompositions of integers into sums of three squares. In fact, if the incomplete primitives of § 8, which have been expressly avoided here, are admitted, all of Kronecker's results and more of the same kind due to later writers reappear; and it appears that the 5, 7, 9, 11, 13 square theorems are of the same general nature. To find closer analogues it is only necessary to replace the functions denoting the numbers of representations as sums of 5, 7, $\cdots$ squares by their weight equivalents as determined by H. J. S. Smith in his memoir "On the Orders and Genera of Quadratic Forms Containing More than Three Indeterminates." The re-

* Read before the San Francisco Section of the American Mathematical Society, April 5, 1919.

† The development of these and other relations from the present point of view will be published elsewhere.

sults are most numerous for 9 squares. The recurrences evaporate with the case of 11 squares. Taken together, all form a short natural chapter in arithmetic.

In counting representations we include, as customary, both the order of the squares and the signs of their square roots. Thus the single decomposition of 54 into a sum of 9 squares,

$$54 = 1^2 + 2^2 + 2^2 + 3^2 + 3^2 + 3^2 + 3^2 + 3^2 + 0^2,$$

contributes $2^8 \cdot 9!/2!5! = 387072$ representations.

As constant use is made of Glaisher's work for 2, 4, $\cdots$, 12 squares, we may cover all references to it by this citation: *Quarterly Journal*, 38 (1906–7), pp. 1–68. A convenient précis of the square-theorem results of his paper is given in the *Proceedings of the London Mathematical Society*, (2) 5 (1907), pp. 479–490.

2. All letters m, μ, n, a, b, c, r, s, denote positive integers, different from zero unless explicitly noted to the contrary; m, μ are always odd, and the rest, unless further specified, arbitrary; k is an integer $\geqq 0$. We define $f(n)$ to be primitive if its values may be calculated in finite form from the real divisors of n alone, and consider $f(x) = 0$ (or, if preferred, non-existent), when $x \lessgtr 0$, or when x is not an integer. With these conventions, the respective sums

$$f(n-4) + f(n-16) + f(n-36) + f(n-64) + \cdots,$$

$$f(n-1) + f(n-9) + f(n-25) + f(n-49) + \cdots,$$

consist of only a finite number of terms, and may be written $\Sigma f(n-4a^2)$, $\Sigma f(n-\mu^2)$, the Σ extending only to those values of a, μ that render $n - 4a^2$, $n - \mu^2 > 0$. Similarly for sums of functions of the triangular numbers, 1, 3, 6, 10, $\cdots$, such as

$$f(n-4) + f(n-12) + f(n-24) + f(n-40) + \cdots,$$

which may be denoted by $\Sigma f(n-4t)$, t representing, as always throughout the paper, a triangular number.

By repeated application of the obvious identity

$$\Sigma f(n-a^2) = \Sigma[f(n-\mu^2) + f(n-4a^2)],$$

we get a transformation which frequently is useful:

$$\Sigma f(n-a^2) = \Sigma[f(n-4^s a^2) + \sum_{r=0}^{s-1} f(n-4^r\mu^2)].$$

3. We shall require the primitives: $\zeta_g(n)$, = the sum of the gth powers of all the divisors of n; $\zeta'_g(n)$, $\zeta''_g(n)$ the like for the odd, even divisors

respectively; $\xi_g(n)$, = the excess of the sum of the gth powers of all those divisors of n that are $\equiv 1 \bmod 4$ over the like sum for the divisors $\equiv -1 \bmod 4$; $\xi_g'(n)$, = the excess of sum of the gth powers of all those divisors of n whose conjugates are $\equiv 1 \bmod 4$ over the like sum for the divisors whose conjugates are $\equiv -1 \bmod 4$; $\epsilon(n) = \epsilon(a^2n)$, = 1 or 0 according as n is or is not a square. From these we construct further primitives as required. When $g = 0$, ζ_g, $\cdots$, ξ_g' are written ζ, $\cdots$, ξ' respectively, and denote the numbers of divisors in the respective classes defined by the corresponding functions. Our notation follows Liouville's, to conform with other papers on similar topics. Glaisher's notation is given loc. cit., pp. 3–6.

4. If $f(n)$ is primitive, then a sum of the form $\Sigma f[(pn - qa^2)/g]$, in which p, q, g are numerical constants, n is an integer, and the summation is with respect to the variable integer a, is defined to be simple. Thus, $\Sigma f(n - a^2)$ is a simple sum, and its value being determined when n is given, we shall call $\Sigma f(n - a^2)$ a simple function of n; and, by a legitimate extension, say that any linear function of a finite number of simple functions of n is a simple function of n. The simple functions most frequently occurring hereafter are of the forms $\Sigma f[(m - \mu^2)/g]$, where $g = 1, 2, 4$ or 8; $\Sigma f(n - a^2)$; $\Sigma f(m - 4a^2)$.

5. Let $N_r(n)$, $N_r(n, g)$ denote respectively the total number of representations of n as a sum of r squares, and the total number of representations of n as a sum of r squares precisely g of which are odd. Then clearly $N_r(n)$ is expressible in the form

$$\sum_{g=0}^{r} c_g N_r(n, g),$$

wherein $c_g = 1$ or 0. When the linear form, modulo s, of n is assigned, the constants c_g are known. We shall use $s = 8$; hence, observing that $m^2 \equiv 1 \bmod 8$, $(2a)^2 \equiv 0$ or $4 \bmod 8$, we have for the determination of g,

$$g + \alpha(r - g) \equiv n \bmod 8, \qquad (\alpha = 0, 4); \qquad 0 \leqq g \leqq r.$$

When r is specified we take in this way a census of the possible linear forms modulo 8. For a particular r the census is most readily found by inspection. Thus, for $r = 11$, (cf. the m, n, k-notation in § 2):

$$\begin{aligned}
n &= 4k &&: N_{11}(n) = N_{11}(n, 0) + N_{11}(n, 4) + N_{11}(n, 8),\\
m &= 4k + 1 &&: N_{11}(m) = N_{11}(m, 1) + N_{11}(m, 5) + N_{11}(m, 9),\\
n &= 2m &&: N_{11}(n) = N_{11}(n, 2) + N_{11}(n, 6) + N_{11}(n, 10),\\
m &= 8k + 3 &&: N_{11}(m) = N_{11}(m, 3) + N_{11}(m, 7) + N_{11}(m, 11),\\
m &= 8k + 7 &&: N_{11}(m) = N_{11}(m, 3) + N_{11}(m, 7).
\end{aligned}$$

Hence, for example, knowing $N_{11}(m, 1)$, $N_{11}(m, 5)$, $N_{11}(m, 9)$ when $m \equiv 1$ or $5 \bmod 8$, we can write down the value of $N_{11}(m)$. The census also shows what values of $N_r(n, g)$ vanish; thus, $N_{11}(2m, 8) = 0$, $N_{11}(2m, 4) = 0$.

6. We shall seek to determine all n, g such that

$$N_r(n), \qquad N_r(n, g), \qquad (r = 3, 5, 7, 9, 11, 13)$$

are simple functions of n, and to exhibit a set of simple functions giving the values of $N_r(n)$, $N_r(n, g)$ in these cases. It will appear that there is not a unique set, for the functions in any set may be transformed in many ways by means of the elementary identities existing between the primitives of § 3. On equating different expressions of the same $N_r(n)$ or $N_r(n, g)$ we get, in several instances, rapid recurrences for the successive calculation of the primitives involved; and in all cases the formulas obtained are well adapted to numerical computation. We may state here, reserving full discussion for another occasion, that when linear forms only are considered, $N_r(n)$is simple for no n when $r = 15, 17, 19, 21, 23, 25$, and probably for no odd $r > 25$; the same applies to $N_r(n, g)$; so that the formulas of this paper form a natural class.

7. To compare the expressions through simple functions with the results given by the classical theory, let us consider, from the standpoint of their adaptability to numerical computation, the three following, of which (A) is due to Eisenstein,* (B) to Stieltjes,† and (C) is found in section IV.

$$(A) \qquad \lambda = 8k + 5: \qquad N_5(\lambda) = -112 \sum_{s=1}^{[\lambda/2]} (s|\lambda)s;$$

$$(B) \qquad m = 8k + 5: \qquad N_5(m, 5) = 32\Sigma\zeta_1\left(\frac{m - \mu^2}{4}\right);$$

$$(C) \qquad m = 8k + 5: \qquad N_5(m) = 112\Sigma\zeta_1\left(\frac{m - \mu^2}{4}\right).$$

In (A), λ is divisible by no square > 1; $[x]$ is the greatest integer in x, and $(s|\lambda)$ is the Legendre-Jacobi symbol, $(s|\lambda) = 0$ when s, λ are not relatively prime. In (B), (C), m is unrestricted, and the summation refers, by the conventions of § 2, to all odd μ such that $0 < \mu \leqq [\sqrt{m}]$. Let $m = 133$; the computation by (A) requires as a first step the calculation of the quadratic characters with respect to 133 of the 55 numbers $1, 2, 3, \cdots, 66$ prime to 133 and < 67. For a large λ the s prime to λ would have to be determined in practically the same way as the $(s|\lambda)$, viz., by Eisenstein's or one of the equivalent algorithms for $(s|\lambda)$, which amounts to the

* Eisenstein, Crelle, 35 (1847), p. 368.

† Stieltjes, *Comptes Rendus*, 97 (1883), p. 981.

conversion of the s/λ into continued fractions, so that for λ not factorable by inspection there would be in all $[\lambda/2]$ such calculations. In that all of these may be performed non-tentatively, formulas such as (A) are superior to (B) or (C). But with the aid of a factor table, it would seem that even for λ fairly large, say $\lambda = 10005$, the 50 resolutions into prime factors required by (C) could be performed more expeditiously than the (approximately) 2500 conversions necessary in (A). Again, if it were required to construct tables, we should have the advantage of recurrences (found in section IX), such as

$$m = 8k + 5: \qquad N_5(m) = -14\xi_2\left(\frac{m+1}{2}\right) - \Sigma N_5(m - 8t),$$

and several similar relations between the primitives ξ_2 whereby their computation may be greatly abridged. For $m = 133$, we find from (C),

$$N_5(133) = 112[\zeta_1(33) + \zeta_1(31) + \zeta_1(27) + \zeta_1(21) + \zeta_1(13) + \zeta_1(3)]$$
$$= 112[48 + 32 + 40 + 32 + 14 + 4] = 19040.$$

8. Functions that may be calculated from the divisors of n subject to one or more conditions were called by Hermite incomplete. Formulas analogous to those of this paper, but involving incomplete functions, may be found on starting from Liouville's '*formules générales*,' or from the elliptic theta equivalents of these was was done by Hermite* for $N_5(n)$ and $N_5(m, 5)$. He found

$$m = 8k + 5: \qquad N_5(m, 5) = G_1(n) + 2\Sigma G_1(m - 4a^2),$$

where $G_1(m)$ is defined by $G_1(m) = 4\Sigma(3d + d')$, the Σ extending to all positive integral solutions of $dd' = m$, $d' > 3d$. Incomplete functions have purposely been avoided in the sequel, as they appear to be less well adapted to computation than the primitives. In regard to Liouville's general formulas for which he did not publish proofs, and which, as indicated may be made to yield the results for 3, 5, $\cdots$, 13 squares, it was remarked by Hurwitz† that Stieltje's results possibly follow from some of them. It might seem, then, that all the simple functions should be derived directly from the same source, without the reference to elliptic functions implied by the use of Glaisher's results. But the reduction is only apparent, the origin of Liouville's formulas appearing most naturally in precisely those elliptic function identities that give the square theorems at once.

* Hermite, Oeuvres, IV, p. 238.

† Hurwitz, *Comptes Rendus*, 98 (1884), pp. 504–507. The two theorems which Liouville derives from his general formulas are insufficient for the proof of Stieltjes' results; cf. footnote to Section IV. Liouville did not indicate the connection between his theorems and representations as sums of 5 squares, nor did he carry out his intention of returning to the subject in a separate article.

II. General Formulas.

9. The notation has been explained in §§ 2, 5, and will be used as there given without further references. Summations being with respect to $\mu = 1, 3, 5, \cdots, \mu_1 = \pm 1, \pm 3, \pm 5, \cdots$, and $a = 1, 2, 3, \cdots, a_1 = 0, \pm 1, \pm 2, \pm 3, \cdots$, we have, in the usual notation of the elliptic theta constants,

$$\vartheta_2(q^4) = \Sigma q^{\mu_1^2} = 2\Sigma q^{\mu^2}, \tag{1}$$

$$\vartheta_3(q) = \Sigma q^{a_1^2} = 1 + 2\Sigma q^{a^2}. \tag{2}$$

Let $N'_r(n, s)$ denote the total number of representations of n as a sum of r squares, precisely s of which are odd and occupy the first s places in the representations; then, obviously

$$r!\, N'_r(n, s) = s!\,(r - s)!\, N_r(n, s), \tag{3}$$

$$\vartheta_2^s(q^4)\vartheta_3^{r-s}(q^4) = \sum_{n=1}^{\infty} q^n N'_r(n, s). \tag{4}$$

Consider the following identities, where $r > 1$:

$$\vartheta_2^{s+1}(q^4)\vartheta_3^{r-s-1}(q^4) = \vartheta_2(q^4) \times \vartheta_2^s(q^4)\vartheta_3^{r-s-1}(q^4), \tag{5}$$

$$\vartheta_2^s(q^4)\vartheta_3^{r-s}(q^4) = \vartheta_3(q^4) \times \vartheta_2^s(q^4)\vartheta_3^{r-s-1}(q^4), \tag{6}$$

$$\vartheta_2(q^4)\vartheta_3^{r-1}(q^4) = \vartheta_2(q^4) \times \vartheta_3^{r-1}(q^4), \tag{7}$$

$$\vartheta_3^r(q) = \vartheta_3(q) \times \vartheta_3^{r-1}(q). \tag{8}$$

By (4) the coefficient of q^n in the left member of (5) is $N'_r(n, s + 1)$. On using the second form of (1) for $\vartheta_2(q^4)$ on the right of (5), applying (4) to the second factor, and multiplying the resulting series (which are absolutely convergent), we find $2\Sigma N'_{r-1}(n - \mu^2, s)$ as the coefficient of q^n. Treating (6), (7), (8) similarly, equating coefficients of like powers of q, and using (3) to replace N' by its equivalent N, we find in this way from (5)–(8) the four fundamental identities, $r > 1$:

$$(s + 1)N_r(n, s + 1) = 2r\Sigma N_{r-1}(n - \mu^2, s), \tag{I}$$

$$(r - s)N_r(n, s) = r[N_{r-1}(n, s) + 2\Sigma N_{r-1}(n - 4a^2, s)], \tag{II}$$

$$m = 4k + 1\colon\ N_r(m, 1) = 2r\left[\epsilon(m) + \Sigma N_{r-1}\left(\frac{m - \mu^2}{4}\right)\right], \tag{III}$$

$$N_r(n) = 2\epsilon(n) + N_{r-1}(n) + 2\Sigma N_{r-1}(n - a^2). \tag{IV}$$

Again, from the definitions, $\vartheta_3^r(q^4) = \Sigma q^{4n} N_r(4n, 0)$; whence, on changing q into $\sqrt[4]{q}$,

$$\vartheta_3^r(q) = \Sigma q^n N_r(4n, 0), = \Sigma q^n N_r(n),$$

the last by (2) and the definition of $N_r(n)$ in § 5. Hence

$$(V) \qquad N_r(4n, 0) = N_r(n).$$

10. It follows from (I). since $\mu^2 \equiv 1 \bmod 8$, that if $N_{r-1}(n, s)$ is primitive for $n \equiv g \bmod 8$, then $N_r(n, s+1)$ is simple when $n \equiv g+1 \bmod 8$. Similarly from (II), if $N_{r-1}(n, s)$ is primitive for $n \equiv g \bmod 4$, then $N_r(n, s)$ is simple for $n \equiv g \bmod 4$; and from (III), for $m \equiv 1 \bmod 4$, $N_r(m, 1)$ is simple if $N_{r-1}(n)$ is primitive, since $(m - \mu^2)/4$ may take any odd or even value, viz., it may take the values n; and the case (IV) shows that $N_r(n)$ is simple if $N_{r-1}(n)$ is primitive. Using (V) we get at once from (III), (IV) the corresponding forms for $N_r(m, 1)$ in terms of $N_{r-1}(m - \mu^2, 0)$, and of $N_r(4n, 0)$. Hence in sifting out the simples for a given r, we examine for what linear forms of n modulo 4 or 8, $N_{r-1}(n)$ is primitive, and apply the appropriate formulas of (I)–(V).

11. From (I), (II) we get, the c's being arbitrary constants:

$$(VI) \qquad \sum_i c_i(1+s_i)N_r(n, 1+s_i) = 2r\sum_i c_i[\Sigma N_{r-1}(n-\mu^2, s_i)],$$

$$(VII) \qquad \sum_i c_i(r-s_i)N_r(n, s_i) = r\sum_i c_i[N_{r-1}(n, s_i) + 2\Sigma N_{r-1}(n-4a^2, s_i)].$$

It is a remarkable fact, first stated by Liouville,* that for r odd and $n \equiv 0$ or $2 \bmod 4$, there always exist integers c_i, s_i depending upon r but not upon n such that $\Sigma_i c_i N_{r-1}(n, s_i)$ is primitive. Moreover, the s_i are in arithmetical progression. Suitably choosing the c_i, s_i we may therefore find in certain cases linear functions, viz., the left members of (VI), (VII) that are simple functions of n, although in general the individual $N_r(n, 1+s_i)$, $N_r(n, s_i)$ in the linear functions are neither primitive nor simple. To avoid reproducing the proofs of Liouville's theorems we shall write down the few necessary cases of (VI), (VII) directly from Glaisher's lists, whence they may be found by inspection.

12. The formulas (I)–(IV) may be reversed. In (I) change r into $r+1$, n into $n+1$, and solve for $N_r(n, s)$; in (II) replace r by $r+1$, and solve for $N_r(n, s)$; in (III) put $r+1$ for r, $4n+1$ for m, and solve for $N_r(n)$; and in (IV) change r into $r+1$ and solve for $N_r(n)$; then

$$(I') \qquad 2(r+1)N_r(n, s) = (s+1)N_{r+1}(n+1, s+1) - 2(r+1)\Sigma N_r(n-8t, s),$$

$$(II') \qquad (r+1)N_r(n, s) = (r+1-s)N_{r+1}(n, s) - 2(r+1)\Sigma N_r(n-4a^2, s),$$

$$(III') \qquad 2(r+1)N_r(n) = N_{r+1}(4n+1, 1) - 2(r+1)[\epsilon(4n+1) + \Sigma N_r(n-2t)],$$

$$(IV') \qquad N_r(n) = N_{r+1}(n) - 2\epsilon(n) - 2\Sigma N_r(n-a^2).$$

* *Journal des Math.*, (2) 6 (1861), 2 papers, pp. 233–, 369–. The proofs were not given, nor a method for determining the c_i when r is given. Both were published in *Bull. Amer. Math. Soc.*, Oct., 1919.

Of these, (IV′) shows that if $N_{r+1}(n)$ is primitive, then the values of $N_r(n)$ may be calculated by recurrence from $N_r(n-1)$, $N_r(n-4)$, $N_r(n-9)$, $\cdots$. Similarly for (I′)–(III′); remembering that t always denotes a triangular number. In each case the value of a primitive for one value of the variable has to be calculated in addition to the N_r functions; thus in (IV′) the assumed primitive is $N_{r+1}(n)$.

III. THREE SQUARES.

13. We first take the census (§ 5) for 3 squares:

$$N_3(4k) = N_3(4k, 0); \quad N_3(4k+1) = N_3(4k+1, 1);$$

$$N_3(4k+2) = N_3(4k+2, 2); \; N_3(8k+3) = N_3(8k+3, 3); \; N_3(8k+7) = 0;$$

next listing the known theorems for 2 squares:

(1) $N_2(n) = 4\xi(n)$; (2) $N_2(m) = N_2(m, 1) = 4\xi(m)$;

(3) $N_2(2m) = N_2(2m, 2) = 4\xi(m)$; $\xi(2^a m) = \xi(m)$,

the last from the definitions in § 3. Having taken the census and tabulated the primitives for a given r, as in (1), (2), (3) above, we then consider (I)–(VII) of § 9, or so many of them as may be relevant to the particular r, here 3, and substitute successively $s = 1, 2, 3, \cdots$ until the total number of possible squares is exhausted, examining at each step the legitimate forms for n or m in $N_r(n, s+1)$, $N_r(n, s)$, etc., effecting this by an inspection of the census; and last, by referring to the primitives, such as (1)–(3) above, find the proper form for the right hand member of (I), (II), $\cdots$. Thus, putting $r = 3$, $s = 1$ in § 9 (I), the left becomes $3N_3(n, 2)$; and since by the census $N_3(n, 2)$ exists only when $n = 2m$, we must have on the right terms of the form $N_2(2m - \mu^2, 1)$, whose value, by (2) above, is $4\xi(2m - \mu^2)$, the variable $2m - \mu^2$ being odd and of the form $4k + 1$ as required by (2). Proceeding thus with all of the formulas (I)–(VII), we find the following cases, in which, *as always henceforth*, the integer in the numbering (2.1), (3.1), etc., of the formulas indicates from which of the primitive representations it has been derived, and the first decimal the particular one of (I)–(VII) used in the derivation. The rest of the numbering is self-explanatory.

Case I. $(s+1)N_3(n, s+1) = 6\Sigma N_2(n - \mu^2, s)$.

(2.1) $N_3(2m) = 12\Sigma\xi(2m - \mu^2)$;

(3.1) $m = 8k + 3$: $N_3(m) = 8\Sigma\xi\left(\dfrac{m - \mu^2}{2}\right)$.

Case II. $(3 - s)N_3(n, s) = 3[N_2(n, s) + 2\Sigma N_2(n - 4a^2, s)]$.

Whence, for $s = 1, 2$:

$$m = 4k + 1: \quad N_3(m) = 6[\xi(m) + 2\Sigma\xi(m - 4a^2)]; \tag{2.2}$$

$$N_3(2m) = 12[\xi(m) + 2\Sigma\xi(m - 2a^2)]. \tag{3.2}$$

The special form $4k + 1$ of m restricts only the formula (2.2) with which it is written, having no relation to subsequent formulas; and so in all similar cases.

Case III. $m = 4k + 1$: $N_3(m, 1) = 6\left[\epsilon(m) + \Sigma N_2\left(\frac{m - \mu^2}{4}\right)\right]$.

$$m = 4k + 1: \ N_3(m) = 6\left[\epsilon(m) + 4\Sigma\xi\left(\frac{m - \mu^2}{4}\right)\right]. \tag{1.3}$$

Case IV. $N_3(n) = 2\epsilon(n) + N_2(n) + 2\Sigma N_2(n - a^2)$.

$$N_3(n) = 2[\epsilon(n) + 2\xi(n) + 4\Sigma\xi(n - a^2)]. \tag{1.4}$$

Case V. $N_3(4n) = 2[\epsilon(n) + 2\xi(n) + 4\Sigma\xi(n - a^2)]$.

Changing n into $2n$, applying the identity in § 2 in the form

$$\Sigma\xi(2n - a^2) = \Sigma\xi(2n - \mu^2) + \Sigma\xi(2n - 4a^2),$$

and noting that $\xi(2b) = \xi(b)$, we get, as an alternative to the special case of (1.4) in which n is a multiple of 8;

$$N_3(8n) = 2[\epsilon(2n) + 2\xi(n) + 4\Sigma\xi(2n - \mu^2) + 4\Sigma\xi(n - 2a^2)]; \tag{4}$$

and for $n = m$,

$$N_3(4m) = 2\left[\epsilon(m) + 2\xi(m) + 4\Sigma\xi\left(\frac{m - \mu^2}{2}\right) + 4\Sigma\xi(m - 4a^2)\right]; \tag{5}$$

both of which illustrate the way in which recurrences for the primitives may be written down on comparing with the equivalent forms deduced from the general (1.4).

IV. Five Squares.*

14. The census for 5 squares is

$N_5(4k) = N_5(4k, 0) + N_5(4k, 4)$; $\quad N_5(8k + 1) = N_5(8k + 1, 1)$;

$N_5(2m) = N_5(2m, 2)$; $\quad N_5(4k + 3) = N_5(4k + 3, 3)$;

$N_5(8k + 5) = N_5(8k + 5, 1) + N_5(8k + 5, 5)$.

Write $\lambda_r(n) = [2(-1)^n + 1]\zeta_r'(n)$, whence

$$\lambda_1(2n) = 3\zeta_1'(n), \ \lambda_1(m) = -\zeta_1(m);$$

* From the formulas (2.1), (6.22), (1.1), (7.2) of cases I, II we get Liouville's results (*J. des Math.*, (2), 4, p. 8); but not conversely.

then the primitive cases for 4 squares as given by Glaisher are:

(1) $N_4(2m, 2)=24\zeta_1(m)$; (2) $N_4(4m, 4)=16\zeta_1(m)$;

(3) $N_4(4n, 0)=8(-1)^n\lambda_1(n)=N_4(n)$; (4) $N_4(2^am)=24\zeta_1(m)$;

(5) $N_4(m)=8\zeta_1(m)$; (6) $m=4k+1$, $N_4(m)=N_4(m, 1)$;

(7) $m=4k+3$, $N_4(m)=N_4(m, 3)$.

Case I. $(s+1)N_5(n, s+1) = 10\Sigma N_4(n-\mu^2, s)$.

whence, for $s = 1, 2, 3, 4$:

(6.1) $N_5(2m) = 40\Sigma\zeta_1(2m-\mu^2)$;

(1.1) $m = 4k+3$: $N_5(m) = 80\Sigma\zeta_1\left(\frac{m-\mu^2}{2}\right)$;

(7.1) $N_5(4n, 4) = 20\Sigma\zeta_1(4n-\mu^2)$;

(2.1) $m = 8k+5$: $N_5(m, 5) = 32\Sigma\zeta_1\left(\frac{m-\mu^2}{4}\right)$.

The last was stated by Stieltjes, cf. § 7.

Case II. $(5-s)N_5(n, s) = 5[N_4(n, s) + 2\Sigma N_4(n-4a^2, s)]$.

Putting $s = 1, 2, 3, 4$, we get:

(6.21) $m = 8k+1$: $N_5(m) = 10[\zeta_1(m) + 2\Sigma\zeta_1(m-4a^2)]$;

(6.22) $m = 8k+5$: $N_5(m, 1) = 10[\zeta_1(m) + 2\Sigma\zeta_1(m-4a^2)]$;

(1.2) $N_5(2m) = 40[\zeta_1(m) + 2\Sigma\zeta_1(m-2a^2)]$;

(7.2) $m+4k+3$: $N_5(m) = 20[\zeta_1(m) + 2\Sigma\zeta_1(m-4a^2)]$;

(2.21) $N_5(4m, 4) = 80[\zeta_1(m) + 2\Sigma\zeta_1(m-4a^2)]$;

(2.22) $N_5(8n, 4) = 160\Sigma\zeta_1(2n-\mu^2)$.

Case III. $m = 4k+1$: $N_5(m, 1) = 10\left[\epsilon(m) + \Sigma N_4\left(\frac{m-\mu^2}{4}\right)\right]$.

(3.31) $m = 8k+1$: $N_5(m) = 10\left[\epsilon(m) + 24\Sigma\zeta_1'\left(\frac{m-\mu^2}{8}\right)\right]$,

(3.32) $m = 8k+5$: $N_5(m, 1) = 80\Sigma\zeta_1\left(\frac{m-\mu^2}{2}\right)$.

Cases IV, V. $N_4(4n, 0) = 2\epsilon(n) + N_4(4n, 0) + 2\Sigma N_4(4n-4a^2, 0)$.

(3.4) $N_5(n) = N_5(4n, 0) = 2\epsilon(n) + 8(-1)^n[\lambda_1(n) + 2\Sigma(-1)^a\lambda_1(n-a^2)]$.

15. From the census, $N_5(4k) = N_5(4k, 0) + N_5(4k, 4)$; hence from

(7.1), (3.4) we get an alternative form of $N_5(4n)$ which may be compared with that given by (3.4). Similarly combining the formulas (2. 1), (3.32), we have

$$(8) \qquad m = 8k + 5\colon\ N_5(m) = 112\Sigma\zeta_1\left(\frac{m-\mu^2}{4}\right).$$

V. Seven Squares.

16. The census is

$$N_7(4k) = N_7(4k, 0) + N_7(4k, 4);$$
$$N_7(2m) = N_7(2m, 2) + N_7(2m, 6);$$
$$N_7(8k+7) = N_7(8k+7, 3) + N_7(8k+7, 7);$$
$$N_7(4k+1) = N_7(4k+1, 1) + N_7(4k+1, 5);$$
$$N_7(8k+3) = N_7(8k+3, 3);$$

also

$$\xi_r'(m) = (-1\,|\,m)\xi_r(m),$$

and from Glaisher's lists the primitives for 6 squares are:

(1) $N_6(2m, 2) = 60\xi_2'(m)$; (2) $m = 4k+3 : N_6(m, 3) = -20\xi_2(m)$;

(3) $N_6(4n, 4) = 240\xi_2'(n)$; (4) $m = 4k+3 : N_6(2m, 6) = -8\xi_2(m)$;

(5) $N_6(n) = N_6(4n, 0) = 4[4\xi_2'(n) - \xi_2(n)]$.

Case I. $(s+1)N_7(n, s+1) = 14\Sigma N_6(n-\mu^2, s)$.

Putting $s = 2, 3, 4, 6$, we find:

$$(1.1) \qquad m = 4k+3\colon\ N_7(m, 3) = 280\Sigma\xi_2'\left(\frac{m-\mu^2}{2}\right);$$

$$(1.11) \qquad m = 8k+3\colon\ N_7(m) = 280\Sigma\xi_2\left(\frac{m-\mu^2}{2}\right);$$

$$(1.12) \qquad m + 8k + 7\colon\ N_7(m, 3) = -280\Sigma\xi_2\left(\frac{m-\mu^2}{2}\right);$$

$$(2.1) \qquad N_7(4n, 4) = -70\Sigma\xi_2(4n-\mu^2);$$

$$(3.1) \qquad m = 4k+1\colon\ N_7(m, 5) = 672\Sigma\xi_2'\left(\frac{m-\mu^2}{4}\right);$$

$$(4.1) \qquad m = 8k+7\colon\ N_7(m, 7) = -16\Sigma\xi_2\left(\frac{m-\mu^2}{2}\right).$$

From (1.12), (4.1) we see that for $m \equiv 7 \bmod 8$, $2N_7(m, 3) = 35N_7(m, 7)$. Many such relations may be read off from the lists for $3, \cdots, 13$ squares.

Case II. $(7-s)N_7(n, s) = 7[N_6(n, s) + 2\Sigma N_6(n-4a^2, s)]$.
Putting $s = 2, 3, 4, 6$:

$$(1.2) \qquad N_7(2m, 2) = 84[\xi_2'(m) + 2\Sigma\xi_2'(m-2a^2)];$$

$$(2.21) \quad m = 8k+3: \quad N_7(m) = -35[\xi_2(m) + 2\Sigma\xi_2(m-4a^2)];$$

$$(2.22) \quad m = 8k+7: \quad N_7(m, 3) = -35[\xi_2(m) + 2\Sigma\xi_2(m-4a^2)];$$

$$(3.2) \qquad N_7(4n, 4) = 560[\xi_2'(n) + 2\Sigma\xi_2'(n-a^2)];$$

$$(4.2) \quad m = 4k+3: \quad N_7(2m, 6) = -56[\xi_2(m) + 2\Sigma\xi_2(m-8a^2)].$$

Case III. $m = 4k+1$: $N_7(m, 1) = 14\left[\epsilon(m) + \Sigma N_6\left(\frac{m-\mu^2}{2}\right)\right]$.

$$(5.3) \quad m=4k+1: \; N_7(m, 1)=14\left[\epsilon(m)+4\Sigma\left\{4\xi_2'\left(\frac{m-\mu^2}{4}\right)-\xi_2\left(\frac{m-\mu^2}{4}\right)\right\}\right].$$

Cases IV, V. $N_7(4n, 0) = N_7(n) =$

$$(5.4) \quad 2\epsilon(n) + 4[4\xi_2'(n) - \xi_2(n)] + 8\Sigma[4\xi_2'(n-a^2) - \xi_2(n-a^2)].$$

The equivalent of (5.4) was stated by Stieltjes, *C.R.*, 31 Dec., 1884.

Cases VI, VII. For the first time these enter. From Glaisher's theorems for 6 squares (loc. cit., p. 10), we find on eliminating non-primitives,

$$(7) \qquad m = 4k+1: \; N_6(m, 1) + N_6(m, 5) = 12\xi_2(m).$$

Hence, taking § 11 (VI) in the form

$$(1+s_1)N_7(n, 1+s_1) + (1+s_2)N_7(n, 1+s_2) = 14\Sigma[N_6(n-\mu^2, s_1) + N_6(n-\mu^2, s_2)],$$

and choosing $(s_1, s_2) = (1, 5)$, we get, on referring to the census,

$$(7.6) \qquad m = 4k+1: \; N_7(2m, 2) + 3N_7(2m, 6) = 84\Sigma\xi_2(2m-\mu^2);$$

and from (VII) in the same way,

$$(7.7) \quad m = 4k+1: \; 3N_7(m, 1) + N_7(m, 5) = 42[\xi_2(m) + 2\Sigma\xi_2(m-4a^2)].$$

Combining (7.7), (3.1) or (7.7), (5, 3) we find alternative forms for $N_7(m, 1)$ or $N_7(m, 5)$ ($m \equiv 1 \bmod 4$) respectively; and from (7.6), (1.2):

$$(8) \quad m = 4k+1: \; N_7(2m, 6) = 28[\Sigma\xi_2(2m-\mu^2) - \xi_2(m) - 2\Sigma\xi_2'(m-2a^2)],$$

which is at once expressible in terms of ξ_2' alone on using the identity of § 2.

17. From the lists in § 16 we write down the following additional complete formulas by reference to the census. Some, such as $N_7(4n)$, which are not much better adapted to computation than those given directly by (5.4), have been omitted.

$$(9)\quad m=4k+1:\quad N_7(m)=14\left[\epsilon(m)+64\Sigma\xi_2'\left(\frac{m-\mu^2}{4}\right)-4\Sigma\xi_2\left(\frac{m-\mu^2}{4}\right)\right];$$

$$(10)\quad m=4k+1:\ N_7(2m)=28[\xi_2(m)+\Sigma\xi_2(2m-\mu^2)+4\Sigma\xi_2'(m-2a^2)];$$

$$(11)\quad m=4k+3:\ N_7(2m)=-28[5\xi_2(m)-6\Sigma\xi_2(m-2\mu^2)+10\Sigma\xi_2(m-8a^2)];$$

$$(12)\quad m=8k+7:\quad N_7(m)=-296\Sigma\xi_2\left(\frac{m-\mu^2}{2}\right).$$

VI. NINE SQUARES.

18. The census is

$$N_9(4n)=N_9(4n,0)+N_9(4n,4)+N_9(4n,8),$$
$$m=8k+1:\ N_9(m)=N_9(m,1)+N_9(m,5)+N_9(m,9),$$
$$N_9(2m)=N_9(2m,2)+N_9(2m,6),$$
$$m=8k+5:\ N_9(m)=N_9(m,1)+N_9(m,5),$$
$$m=4k+3:\ N_9(m)=N_9(m,3)+N_9(m,7).$$

To state the primitives for 8 squares it is convenient to introduce $\rho_3(n)$, $\alpha_3(n)$, where $\rho_r(n)=\zeta_r'(n)-\zeta_r''(n)$, and $\alpha_r(n)=n^r\zeta'_{-r}(n)=$ the sum of the rth powers of all those divisors of n whose conjugates are odd. From these definitions we have at once the useful identities:

$$\alpha_3(m)=\zeta_3(m),\quad \alpha_3(2m)=8\zeta_3(m),\quad \alpha_3(2n)=8\alpha_3(n),$$
$$\rho_3(m)=\zeta_3(m),\quad \rho_3(2m)=-7\zeta_3(m),\quad \rho_3(2n)=8\rho_3(n)-15\zeta_3'(n),$$
$$7\rho_3(n)+8\alpha_3(n)=15\zeta_3'(n).$$

The primitives for 8 squares, from Glaisher, are:

(1) $N_8(4n,4)=1120\alpha_3(n)$; (2) $N_8(8n,8)=256\alpha_3(n)$;

(3) $N_8(n)=-16(-1)^n\rho_3(n)$; (4) $N_8(4n,0)=N_8(n)$.

Case I. $(s+1)N_9(n,s+1)=18\Sigma N_8(n-\mu^2,s)$.

Whence, for $s=4, 8$:

$$(1.1)\quad m=4k+1:\ N_9(m,5)=4032\Sigma\alpha_3\left(\frac{m-\mu^2}{4}\right);$$

$$(1.11)\quad m=8k+1:\ N_9(m,5)=32256\Sigma\alpha_3\left(\frac{m-\mu^2}{8}\right);$$

$$(1.12)\quad m=8k+5:\ N_9(m,5)=4032\Sigma\zeta_3\left(\frac{m-\mu^2}{4}\right);$$

$$(2.1)\quad m=8k+1:\ N_9(m,9)=512\Sigma\alpha_3\left(\frac{m-\mu^2}{8}\right).$$

Case II. $(9-s)N_9(n,s) = 9[N_8(n,s) + 2\Sigma N_8(n-4a^2, s)]$.

Putting $s = 4, 8$, and separating the cases according as $m \equiv 0$ or $4 \bmod 8$, we find, after some short reductions:

(1.2) $N_9(4n, 4) = 2016[\alpha_3(n) + 2\Sigma\alpha_3(n-a^2)]$;

(1.21) $N_9(4m, 4) = 2016\left[\zeta_3(m) + 2\Sigma\zeta_3(m-4a^2) + 16\Sigma\alpha_3\left(\frac{m-\mu^2}{2}\right)\right]$;

(1.22) $N_9(8n, 4) = 4032[4\alpha_3(n) + 8\Sigma\alpha_3(n-2a^2) + \Sigma\zeta_3(2n-\mu^2)]$;

(2.21) $N_9(4m, 8) = 4608\Sigma\alpha_3\left(\frac{m-\mu^2}{2}\right)$;

(2.22) $N_9(8n, 8) = 2304[\alpha_3(n) + 2\Sigma\alpha_3(n-2a^2)]$.

Case III. $m = 4k+1$: $N_9(m, 1) = 18\left[\epsilon(m) + \Sigma N_8\left(\frac{m-\mu^2}{4}\right)\right]$.

(3.31) $m = 8k+1$: $N_9(m, 1)$

$$= 18\left[\epsilon(m) + 240\Sigma\zeta_3'\left(\frac{m-\mu^2}{8}\right) - 128\Sigma\rho_3\left(\frac{m-\mu^2}{8}\right)\right];$$

(3.32) $m = 8k+5$: $N_9(m, 1) = 288\Sigma\zeta_3\left(\frac{m-\mu^2}{4}\right)$.

Cases IV, V. $N_9(n) = N_9(4n, 0) =$

(3.4) $2\epsilon(n) - 16(-1)^n[\rho_3(n) + 2\Sigma(-1)^a\rho_3(n-a^2)]$.

Cases VI, VII. From Glaisher's results for 8 squares we have:

(5) $m = 4k+1$: $N_8(m, 1) + N_8(m, 5) = 16\zeta_3(m)$;

(6) $N_8(2m, 2) + N_8(2m, 6) = 112\zeta_3(m)$;

(7) $m = 4k+3$: $N_8(m, 3) + N_8(m, 7) = 16\zeta_3(m)$.

Hence, putting $(s_1, s_2) = (1, 5), (2, 6), (3, 7)$ in (VI), (VII) of § 11, and comparing as usual with the census and primitives:

(5.6) $N_9(2m, 2) + 3N_9(2m, 6) = 144\Sigma\zeta_3(2m-\mu^2)$;

(6.6) $m = 4k+3$: $3N_9(m, 3) + 7N_9(m, 7) = 2016\Sigma\zeta_3\left(\frac{m-\mu^2}{2}\right)$;

(7.6) $N_9(4n, 4) + 2N_9(4n, 8) = 72\Sigma\zeta_3(4n-\mu^2)$;

(5.7) $m = 4k+1$: $2N_9(m, 1) + N_9(m, 5) = 36[\zeta_3(m) + 2\Sigma\zeta_3(m-4a^2)]$;

$$(6.7) \qquad 7N_9(2m, 2) + 3N_9(2m, 6) = 1008[\zeta_3(m) + 2\Sigma\zeta_3(m - 2a^2)];$$

$$(7.7) \quad m = 4k + 3: \; 3N_9\,(m, 3) + N_9\,(m, 7) = 72[\zeta_3(m) + 2\Sigma\zeta_3(m - 4a^2)].$$

Solving these we get:

$$(9) \qquad N_9(2m, 2) = 24[7\zeta_3(m) + 14\Sigma\zeta_3(m - 2a^2) - \Sigma\zeta_3(2m - \mu^2)];$$

$$(10) \qquad N_9(2m, 6) = 56[-\zeta_3(m) - 2\Sigma\zeta_3(m - 2a^2) + \Sigma\zeta_3(2m - \mu^2)];$$

$$(11) \quad m = 4k + 3:$$

$$N_9\,(m, 3) = 28\left[\zeta_3(m) + 2\Sigma\zeta_3(m - 4a^2) - 4\Sigma\zeta_3\left(\frac{m - \mu^2}{2}\right)\right];$$

$$(12) \quad m = 4k + 3:$$

$$N_9\,(m, 7) = 12\left[-\zeta_3(m) - 2\Sigma\zeta_3(m - 4a^2) + 28\Sigma\zeta_3\left(\frac{m - \mu^2}{2}\right)\right].$$

Similarly from (1.1), (5.7):

$$(13) \quad m = 4k + 1:$$

$$N_9(m, 1) = 18\left[\zeta_3(m) + 2\Sigma\zeta_3(m - 4a^2) - 112\Sigma\alpha_3\left(\frac{m - \mu^2}{4}\right)\right].$$

19. Combining certain of the formulas in § 18 according to the census we find the additional complete cases in the following list. Thus, the value of $N_9(2m) = N_9(2m, 2) + N_9(2m, 6)$ follows from (9), (10). Some that may be found in this way have been omitted.

$$(14) \quad m = 8k + 1:$$

$$N_9\,(m) = 18\zeta_3(m) + 36\Sigma\zeta_3(m - 4a^2) + 16640\Sigma\alpha_3\left(\frac{m - \mu^2}{8}\right);$$

$$(15) \quad m = 8k + 5:$$

$$N_9\,(m) = 4320\Sigma\zeta_3\left(\frac{m - \mu^2}{4}\right);$$

$$(16) \quad m = 4k + 3:$$

$$N_9\,(m) = 16\left[\zeta_3(m) + 2\Sigma\zeta_3(m - 4a^2) + 14\Sigma\zeta_3\left(\frac{m - \mu^2}{2}\right)\right];$$

$$(17) \quad N_9(2m) = 16[7\zeta_3(m) + 14\Sigma\zeta_3(m - 2a^2) + 2\Sigma\zeta_3(2m - \mu^2)].$$

VII. ELEVEN SQUARES.*

20. The census is in § 5; and the primitives for ten squares were found by Glaisher to be:

* The references in Bachmann's Zahlentheorie, Bd. IV, and the German Encyclopaedia to Liouville's consideration of this case are misprints; Liouville published nothing concerning 11 squares.

(1) $5N_{10}(n) = 4[\xi_4(n) + 16\xi_4'(n)]$; (2) $N_{10}(2m, 2) = -\ 36\xi_4(m)$;

(3) $N_{10}(4n, 4) = 2688\xi_4'(n)$; (4) $N_{10}(2m, 6) = -\ 168\xi_4(m)$;

(5) $N_{10}(4n, 8) = 576\xi_4'(n)$; (6) $N_{10}(4n, 0) = N_{10}(n)$,

in all of which $n = 2^b m$, $m = 4k + 3$, $b \geqq 0$. For the same value of n, $\xi_4'(n) = -\ 2^{4b}\xi_4(m)$, $\xi_4(n) = \xi_4(m)$; and the numbers $n - a^2$ ($a = 1, 2, 3, \cdots$) are all of the form $2^c(4k + 3)$, $c \geqq 0$, when and only when $b = 0, 2$, and $m = 8k + 7$. Hence we find only the following simple expressions for 11 squares.

Case I. $(s + 1)N_{11}(n, s + 1) = 22\Sigma N_{10}(n - \mu^2, s)$.

Whence, for $s = 2, 6$:

$$(2.1) \qquad m = 8k + 7\text{: } N_{11}(m, 3) = -\ 264\Sigma\xi_4\left(\frac{m - \mu^2}{2}\right);$$

$$(4.1) \qquad m = 8k + 7\text{: } N_{11}(m, 7) = -\ 528\Sigma\xi_4\left(\frac{m - \mu^2}{2}\right).$$

Case II. $(11 - s)N_{11}(n, s) = 11[N_{10}(n, s) + 2\Sigma N_{10}(n - 4a^2, s)]$.

Comparing this with (3), (5), we see that when and only when n is of either form $8k + 7$, $4(8k + 7)$,

$$(3.2) \qquad N_{11}(4n, 4) = 4224[\xi_4'(n) + 2\Sigma\xi_4'(n - a^2)];$$

$$(5.2) \qquad N_{11}(4n, 8) = 2112[\xi_4'(n) + 2\Sigma\xi_4'(n - a^2)].$$

Case IV. $N_{11}(n) = 2\epsilon(n) + N_{10}(n) + 2\Sigma N_{10}(n - a^2)$.

Hence for n as in case II, and $m = 8k + 7$:

$$(1.4) \qquad 5N_{11}(n) = 4[\xi_4(n) + 16\xi_4'(n) + 2\Sigma\{\xi_4(n - a^2) + 16\xi_4'(n - a^2)\}];$$

$$(1.41) \qquad N_{11}(m) = -\ 12\left[\xi_4(m) + 2\Sigma\xi_4(m - 4a^2) + 32\Sigma\xi_4\left(\frac{m - \mu^2}{2}\right)\right].$$

21. Combining (2.1), (4.1) by the census,

$$(7) \qquad m = 8k + 7\text{: } N_{11}(m) = -\ 792\Sigma\xi_4\left(\frac{m - \mu^2}{2}\right).$$

The census gives also $N_{11}(4m) = N_{11}(4m, 0) + N_{11}(4m, 4) + N_{11}(4m, 8)$, and (1.4) the value of $N_{11}(4m, 0)$; hence from these and (3.2), (5.2) may be derived recurrences for the ξ_4, ξ_4'; and similarly from (1.4) and (7).

22. By eliminating the non-primitives from the formulas for $N_{10}(n, r)$ as given by Glaisher, it is easy to deduce several relations of the forms (VI), (VII) of § 11, and therefore to find simple linear functions of N's. But, as in preceding sections, it is verified without difficulty that the com-

plete system of such relations when combined with the formulas of §§ 20, 21 is insufficient for the determination of any N_{11} not already listed. This verification, presenting nothing new, is omitted. The like may be shown to hold for 13, 15, 17, 19, 21, 23, 25 squares; and when we pass to 13 squares (the next case considered) there is the concurrent disappearance of primitives for the associated even number of squares. Whether both or either of these failures occurs always for m squares when > 25, has been shown in another paper to depend upon the divisors common to certain binomial coefficients and the numerators of the numerical coefficients in the power series for the Jacobian elliptic functions, and need only be mentioned here. We remark, however, that one aspect of the failure is permanent after 13 squares; viz., the number of linear relations between the several $N_m(n, r)$ is always less than the number of unknown functions.

VIII. THIRTEEN SQUARES.

23. It is unnecessary to take the census. $\alpha_r(n)$ is defined in § 18; and we now introduce $\beta_r(n)$, $= \zeta'_r(n) - \zeta''_r(n) + 2\alpha_r(n)$. This function is susceptible of many transformations; in particular it may be expressed in terms of 2^s and $\zeta_r(m)$, where $n = 2^s m$, but the form stated is sufficient for our purpose. The only primitives for 12 squares are

$$(1)\ N_{12}(2n) = 8\beta_5(2n); \qquad (2)\ N_{12}(8n, 4) = N_{12}(8n, 8) = 126720\alpha_5(n).$$

An equivalent form of (1) was stated with insufficient proof by Liouville,* and first proved by Glaisher, from whose paper both are transcribed. We shall consider the primitive $N_{12}(8n, 0)$ as included under (1). It is readily seen that Cases I, III alone are applicable.

Case I. $\qquad (s+1)N_{13}(n, s+1) = 26\Sigma N_{12}(n - \mu^2, s).$

Whence, for $s = 4, 8$:

$$(2.11)\qquad m = 8k+1:\ N_{13}(m, 5) = 658944\Sigma\alpha_5\left(\frac{m-\mu^2}{8}\right);$$

$$(2.12)\qquad m = 8k+1:\ N_{13}(m, 9) = 366080\Sigma\alpha_5\left(\frac{m-\mu^2}{8}\right).$$

Hence $5N_{13}(m, 5) = 9N_{13}(m, 9)$ when $m \equiv 1 \bmod 8$.

* *Journal des Math.*, (2) 6 (1861), p. 206. The proof is insufficient because it is made to depend upon a theorem which Liouville did not prove, and whose proof is more difficult than that of the special result for 12 squares; cf. Bachmann, Zahlentheorie, Bd. IV, p. 668. The same applies also to the primitives for 10 squares, first stated by Eisenstein and Liouville, but first proved by Glaisher.

Case III. $m = 4k + 1$: $N_{13}(m, 1) = 26\left[\epsilon(m) + \Sigma N_{12}\left(\frac{m - \mu^2}{4}\right)\right]$.

(1.3) $m = 8k + 1$: $N_{13}(m, 1) = 26\left[\epsilon(m) + 8\Sigma\beta_5\left(\frac{m - \mu^2}{4}\right)\right]$.

24. The census gives

$$m = 8k + 1\text{: } N_{13}(m) = N_{13}(m, 1) + N_{13}(m, 5) + N_{13}(m, 9).$$

Hence, from § 23, we have

(3) $m = 8k + 1$:

$$N_{13}(m) = 26\left[\epsilon(m) + 8\Sigma\beta_5\left(\frac{m - \mu^2}{4}\right) + 39424\Sigma\alpha_5\left(\frac{m - \mu^2}{8}\right)\right].$$

IX. RECURRENCES.

25. In the numbering (3.1), etc., of the following formulas the integer $= r$, and the first decimal $= s$, and the resulting recurrence is written down from the general formula at the head of its set by substituting the values of (r, s) thus defined. *E.g.*, we get the first (3.1) on putting $r = 3$ $s = 1$ in (I′), and referring to § 14 for the value of

$$(s + 1)N_{r+1}(m + 1, s + 1), = 2N_4(m + 1, 2),$$

noting that $m + 1$ is an even integer, say $2n$; hence $m = 2n - 1$, and therefore being any odd positive integer, is denoted by m in (3.1). Similarly all the recurrences under the several cases (I′)–(IV′) are written down by inspection on glancing first at the primitives for $N_{r+1}(m + 1, s + 1)$, $N_{r+1}(n, s)$, $N_{r+1}(4n + 1, 1)$, $N_{r+1}(n)$ respectively, and giving r the values 3, 5, 7, 9, 11, in succession. We recall that t represents always a triangular number. Cases I′–IV′ are from § 12.

Case I′. $2(r + 1)N_r(n, s) = (s + 1)N_{r+1}(n + 1, s + 1) - 2(r + 1)\Sigma N_r(n - 8t, s)$.

(3.1) $m = 4k + 1$: $N_3(m) = 6\zeta_1\left(\frac{m + 1}{2}\right) - \Sigma N_3(m - 8t)$;

(3.2) $N_3(2m) = 3\zeta_1(2m + 1) - \Sigma N_3(2m - 8t)$;

(3.3) $m = 8k + 3$: $N_3(m) = 8\zeta_1\left(\frac{m + 1}{4}\right) - \Sigma N_3(m - 8t)$;

(5.11) $m = 8k + 1$: $N_5(m) = 10\xi_2\left(\frac{m + 1}{2}\right) - \Sigma N_5(m - 8t)$;

(5.12) $m = 8k + 5$: $N_5(m, 1) = -10\xi_2\left(\frac{m + 1}{2}\right) - \Sigma N_5(m - 8t, 1)$;

(5.2) $$N_5(2m) = -5\xi_2(2m+1) - \Sigma N_5(2m-8t);$$

(5.3) $m = 4k+3$: $$N_5(m) = 80\xi_2'\left(\frac{m+1}{4}\right) - \Sigma N_5(m-8t);$$

(5.5) $m = 8k+5$: $$N_5(m,5) = -4\xi_2\left(\frac{m+1}{2}\right) - \Sigma N_5(m-8t,5);$$

(5.51) $m = 8k+5$: $$N_5(m) = -14\xi_2\left(\frac{m+1}{2}\right) - \Sigma N_5(m-8t);$$

(7.31) $m = 8k+3$: $$N_7(m) = 280\alpha_3\left(\frac{m+1}{4}\right) - \Sigma N_7(m-8t);$$

(7.32) $m = 8k+7$: $$N_7(m,3) = 2240\alpha_3\left(\frac{m+1}{8}\right) - \Sigma N_7(m-8t,3);$$

(7.7) $m = 8k+7$: $$N_7(m,7) = 128\alpha_3\left(\frac{m+1}{8}\right) - \Sigma N_7(m-8t,7);$$

(7.71) $m = 8k+7$: $$N_7(m) = 2368\alpha_3\left(\frac{m+1}{8}\right) - \Sigma N_7(m-8t);$$

(9.1) $m = 8k+5$: $$5N_9(m,1) = -18\xi_4\left(\frac{m+1}{2}\right) - 5\Sigma N_9(m-8t,1);$$

(9.31) $m = 16k+11$: $$5N_9(m,3) = 2688\xi_4'\left(\frac{m+1}{4}\right) - 5\Sigma N_9(m-8t,3);$$

(9.5) $m = 8k+5$: $$5N_9(m,5) = -252\xi_4\left(\frac{m+1}{2}\right) - 5\Sigma N_9(m-8t,5);$$

(9.51) $m = 8k+5$: $$N_9(m) = -54\xi_4\left(\frac{m+1}{2}\right) - \Sigma N_9(m-8t);$$

(9.7) $m = 16k+11$: $$5N_9(m,7) = 1152\xi_4'\left(\frac{m+1}{4}\right) - 5\Sigma N_9(m-8t,7);$$

(9.71) $m = 16k+11$: $$N_9(m) = 768\xi_4'\left(\frac{m+1}{4}\right) - \Sigma N_9(m-8t);$$

(11.3) $m = 8k+7$: $$N_{11}(m,3) = 21120\alpha_5\left(\frac{m+1}{8}\right) - \Sigma N_{11}(m-8t,3);$$

(11.7) $m = 8k+7$: $$N_{11}(m,7) = 44240\alpha_5\left(\frac{m+1}{8}\right) - \Sigma N_{11}(m-8t,7);$$

(11.71) $m = 8k+7$: $$N_{11}(m) = 63360\alpha_5\left(\frac{m+1}{8}\right) - \Sigma N_{11}(m-8t).$$

Case II'. $(r+1)N_r(n,s) = (r+1-s)N_{r+1}(n,s) - 2(r+1)\Sigma N_r(n-4a^2, s).$

(3.1) $m = 4k+1$: $N_3(m) = 6\zeta_1(m) - 2\Sigma N_3(m-4a^2);$

(3.2) $N_3(2m) = 12\zeta_1(m) - 2\Sigma N_3(2m-4a^2);$

(3.3) $m = 8k+3$: $N_3(m) = 2\zeta_1(m) - 2\Sigma N_3(m-4a^2);$

(5.2) $N_5(2m) = 40\xi_2'(m) - 2\Sigma N_5(2m-4a^2);$

(5.3) $m = 4k+3$: $N_5(m) = -10\xi_2(m) - 2\Sigma N_5(m-4a^2);$

(5.4) $N_5(4n,4) = 80\xi_2'(n) - 2\Sigma N_5(4n-4a^2, 4);$

(7.4) $N_7(4n,4) = 560\alpha_3(n) - 2\Sigma N_7(4n-4a^2, 4);$

(9.2) $m = 4k+3$: $5N_9(2m,2) = -144\xi_4(m) - 10\Sigma N_9(2m-4a^2, 2);$

(9.4) $n = 2^b(4k+3)$, $b \geqq 0$:

$5N_9(4n,4) = 8064\xi_4'(n) - 10\Sigma N_9(4n-4a^2, 4);$

(9.6) $m = 4k+3$: $5N_9(2m,6) = -336\xi_4(m) - 10\Sigma N_9(2m-4a^2, 6);$

(9.61) $m = 4k+3$: $N_9(2m) = -96\xi_4(m) - 10\Sigma N_9(2m-4a^2);$

(9.8) $n = 2^b(4k+3)$, $b \geqq 0$:

$5N_9(4n,8) = 576\xi_4'(n) - 10\Sigma N_9(4n-4a^2, 8);$

(11.4) $N_{11}(8n,4) = 84480\alpha_5(n) - 2\Sigma N_{11}(8n-4a^2, 4);$

(11.8) $N_{11}(8n,8) = 42240\alpha_5(n) - 2\Sigma N_{11}(8n-4a^2, 8).$

Case III'. $2(r+1)N_r(n) = N_{r+1}(4n+1, 1) - 2(r+1)[\epsilon(4n+1) + \Sigma N_r(n-2t)].$

(3.1) $N_3(n) = \zeta_1(4n+1) - \epsilon(4n+1) - \Sigma N_3(n-2t).$

Case IV'. $N_r(n) = N_{r+1}(n) - 2\epsilon(n) - 2\Sigma N_r(n-a^2).$

(3) $N_3(n) = 8(-1)^n\lambda_1(n) - 2\epsilon(n) - 2\Sigma N_3(n-a^2);$

(5) $N_5(n) = 4[\xi_2'(n) - \xi_2(n)] - 2\epsilon(n) - 2\Sigma N_5(n-a^2);$

(7) $N_7(n) = -16(-1)^n\rho_3(n) - 2\epsilon(n) - 2\Sigma N_7(n-a^2);$

(9) $n = 2^b(4k+3)$, $b \geqq 0$:

$5N_9(n) = 4[\xi_4(n) + 16\xi_4'(n)] - 10\epsilon(n) - 10\Sigma N_9(n-a^2);$

(11) $N_{11}(2n) = 8\beta_5(n) - 2\epsilon(2n) - 2\Sigma N_{11}(2n-a^2).$

Since $n - a^2$ is not always of the form $2^b(4k+3)$, (9) does not enable us

to calculate the values of $N_9(n)$ by recurrence alone; and the same holds for (11), half the values of $2n - a^2$ being odd. The like applies to certain of the formulas under I′–III′; but in conjunction with the results of sections III–VII, even the incomplete recurrences are a material aid to computation.

26. Comparing the finite sums given by $N_r(n)$ under cases IV in sections III–VII with those determined under cases I–III, we may derive useful recurrences for certain of the primitives. These, which are of the same general type as the two examples given by Liouville (cf. § 14, footnote), we omit.

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ON A CERTAIN CLASS OF RATIONAL RULED SURFACES.

By Arnold Emch.

1. Introduction.

As is well known, ruled surfaces may be generated or defined in a number of ways. There exists, for example, a one-to-one correspondence between ruled surfaces, or scrolls as Cayley calls them, and a certain class of partial differential equations, so that the theories of the two classes are abstractly equivalent.

A much favored method, especially in descriptive geometry, consists in considering ruled surfaces as continuous sets of straight lines (generatrices), which intersect three fixed curves, (the directrices), simultaneously. If these are algebraic curves of orders l, m, n, with no points in common, the ruled surface which they define is, in general, of order $2lmn$.

Frequently ruled surfaces are also defined as systems of elements, either common to two rectilinear congruences, or to three rectilinear complexes.

Of great importance for the following investigation is the definition of ruled surfaces as systems of lines which join corresponding points of an (α, β)-correspondence between the points of two algebraic curves C_m and C_n of orders m and n. If the curves are plane, and if to a point of C_m correspond α points of C_n, and to a point of C_n β points of C_m, then the order of the surface is, in general, $\alpha m + \beta n$.

Finally there is the cinematic method in which ruled surfaces are generated by the continuous movement of the generatrix according to some definite cinematical law. In this connection the description of the hyperboloid of rotation of one sheet by a straight line rotating about a fixed axis is well known. The literature seems to contain but little about this method of generating ruled surfaces.* A number of treatises on differential geometry contain chapters on cinematically generated surfaces.†

In this paper the results of an investigation of a rather extended class of ruled surfaces are presented, which are defined cinematically. The class is

* E. M. Blake, "Two Plane Movements Generating Quartic Scrolls," *Transactions of the American Mathematical Society*, Vol. 1, pp. 421–429 (1900).

† Darboux, "Leçons sur la théorie générale des surfaces," Vol. 1, 2d ed., pp. 127–150 (1914).

Eisenhart, "A Treatise on the Differential Geometry of Curves and Surfaces," pp. 146–148 (1909).

obtained as follows: Given a directrix circle C_2 and a directrix line C_1, which passes through the center of C_2 and is at right angles to the plane of C_2. The generatrix g moves so that a fixed point M of g moves along C_2 uniformly, while g in every position passes through C_1. The plane e through C_1 in which g lies evidently rotates about C_1 with the same velocity $k\theta$ as M. In this plane e, the generatrix g rotates at the same time with uniform velocity $\mu k\theta$ about M. It will be shown that g generates a rational algebraic surface of the class when $\mu = p/q$ is a rational fraction, and that this class thus described is equivalent to a class of ruled surfaces obtained by means of an (α, β)-correspondence between C_1 and C_2, in which α and β will be determined hereafter.

Among the most important references bearing more closely upon the subject may be mentioned papers by Cremona,* Armamente,† Cayley,‡ Schwarz,§ Noether,|| Clebsch,¶ Picard.**

2. Parametric Equations and Order of the Surface.

Let C_1 coincide with the z-axis, so that C_2 lies in the xy-plane, and g any position of the generatrix, P any point on g, P' its projection upon the xy-plane, and M the intersection of g with C_2. By ρ denote the radius vector OP', whose direction forms an angle θ with the positive part of the x-axis. In the plane e through C_1, in which g lies, g is determined by the angle ψ which g makes with the positive direction of the perpendicular through M to the xy-plane, ψ being measured in the clock-wise sense. By the angles θ and ψ the position of g is perfectly determined. Assuming $\psi = 0$, when $\theta = 0$, and imposing upon ψ the condition $\psi = p/q \cdot \theta$, where p and q are relatively prime integers, the surface of the class, characterized by two definite values of p and q, may be represented parametrically by

$$x = \rho \cos\theta, \qquad y = \rho \sin\theta, \qquad z = (\rho - a) \cot\frac{p}{q}\theta, \tag{1}$$

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* "Rappresentazione di una classe di superficie gobbe sopra un piano, e determinazione delle loro curve assintotiche," *Annali di Matematica*, Ser. II, Vol. 1, pp. 248–258 (1867).

† "Intorno alla rappresentazione della superficie gobbe di genere $p = 0$," *Annali di Matematica*, Ser. II, Vol. 4, pp. 50–72 (1870).

‡ "On Certain Skew Surfaces, otherwise Scrolls," *Transactions of the Cambridge Philosophical Society*, Vol. XI, Part II, pp. 277–289 (1869).

§ "Ueber die geradlinigen Flächen fünften Grades," *Journal für reine und angewandte Mathematik*, Vol. 76, pp. 23–57 (1867).

|| "Ueber Flächen, welche Schaaren rationaler Curven besitzen," *Mathematische Annalen*, Vol. 3, pp. 161–227 (1871).

¶ "Ueber die geradlinigen Flächen vom Geschlechte $p = 0$," *Mathematische Annalen* Vol. 5, pp. 1–26 (1872).

** "Sur les surfaces algébriques dont toutes les sections planes sont unicursales," *Journal für reine und angewandte Mathematik*, Vol. 100, pp. 71–78 (1885).

in which a is the radius of C_2, ρ and θ are the parametric coördinates of a point of the surface. Since

$$\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{1-\tan^2\left(q\cdot\frac{\theta}{2q}\right)}{1+\tan^2\left(q\cdot\frac{\theta}{2q}\right)},$$

$$\sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{2\cdot\tan\left(q\cdot\frac{\theta}{2q}\right)}{1+\tan^2\left(2q\frac{\theta}{q\cdot}\right)},$$

$$\cot\frac{p}{q}\theta = 1/\tan\left(2p\cdot\frac{\theta}{2q}\right),$$

may all be expressed as rational functions of the parameter $\tan\frac{\theta}{2q} = t$, x, y, z become rational functions of the parameters ρ and t, so that (1) *defines a rational ruled surface.*

To determine the order of the surface, the number of points of intersection of the line

$$a_1x = b_1y + c_1z + d_1 = 0, \qquad a_2x + b_2y + c_2z + d_2 = 0 \tag{2}$$

with the surface must be found. Substituting in (2) for x, y, z their expressions given by (1), and eliminating ρ, the equation

$$\{(a_1d_2 - a_2d_1)\cos\theta + (b_1d_2 - b_2d_1)\sin\theta\}\tan\frac{p}{q}\theta$$
$$-(a_1c_2 - a_2c_1)\cos\theta - (b_1c_2 - b_2c_1)\sin\theta + c_1d_2 - c_2d_1 = 0 \tag{3}$$

is obtained. Making use of the identities

$$1 + \tan^2 m\theta = \cos^{2m}\theta(1+\tan^2\theta)^m/\cos^2 m\theta, \tag{4}$$

$$\sin m\alpha = \binom{m}{1}\cos^{m-1}\alpha\cdot\sin\alpha - \binom{m}{3}\cos^{m-3}\alpha\cdot\sin^3\alpha + \cdots, \tag{5}$$

$$\cos m\alpha = \binom{m}{0}\cos^m\alpha - \binom{m}{2}\cos^{m-2}\alpha\cdot\sin^2\alpha + \cdots, \tag{6}$$

and putting $\frac{\theta}{2q} = \alpha$, $\tan\alpha = t$,

$$\frac{\sin q\alpha}{\cos^q\alpha} = \binom{q}{1}\tan\alpha - \binom{q}{3}\tan^3\alpha + \cdots \pm \tan^{-q}\alpha = \phi^q(t),$$

$$\frac{\cos q\alpha}{\cos^q\alpha} = \binom{q}{0} - \binom{q}{2}\tan^2\alpha + \cdots \pm \binom{q}{q-1}\tan^{q-1}\alpha = \psi^{q-1}(t),$$

where ϕ and ψ are rational integral functions of t of order q and $q-1$, as indicated by the upper indices. In this manner we obtain for $\sin\theta$, $\cos\theta$, $\tan\frac{p}{q}\theta$ the expressions,

$$\sin\theta = \frac{2\phi^q(t)\psi^{q-1}(t)}{(1+t^2)^q},\quad \cos\theta = \frac{\psi^{2q-2}(t)-\phi^{2q}(t)}{(1+t^2)^q},\quad \tan\frac{p}{q}\theta = \frac{2\phi_1(t)\cdot\psi_1(t)}{\psi_1^2(t)-\phi_1^2(t)}. \tag{7}$$

The numerator of the expression for $\tan(p/q)\theta$ is of degree $2p-1$, the denominator of degree $2p$ in t. Substituting these expressions in (5), an equation of degree $2(p+q)$ in t is obtained, so that when q is odd, a line (2) cuts the surface in as many points. Hence, *when q is odd, the order of the surface is $2(p+q)$.*

When $q=2s$ is even, so that in (5)

$$\tan\frac{p}{q}\theta = \tan\left(p\cdot\frac{\theta}{2s}\right),$$

then putting $\tan(\theta/2s)=t$, with p odd, there is, in analogy with (7),

$$\sin\theta = \frac{2\tan\left(s\cdot\frac{\theta}{2s}\right)}{1-\tan^2\left(s\cdot\frac{\theta}{2s}\right)} = \frac{2\phi^s(t)\cdot\psi^{s-1}(t)}{(1+t^2)^s},$$

$$\cos\theta = \frac{1-\tan^2\left(s\cdot\frac{\theta}{2s}\right)}{1+\tan^2\left(s\cdot\frac{\theta}{2s}\right)} = \frac{\psi^{2s-2}(t)-\phi^{2s}(t)}{\psi^{2s-2}(t)+\phi^{2s}(t)},$$

$$\tan\frac{p}{q}\theta = \tan\left(p\cdot\frac{\theta}{2s}\right) = \frac{\psi^p(t)}{\psi^{p-1}(t)},$$

in which the upper indices indicate again the degrees of the polynomials ϕ and ψ in t. Substituting these expressions in (3), an equation of degree $p+q$ in t is obtained, so that in this case the order of the surface is $p+q$. The results may be stated as

THEOREM 1. *The surface of the class is rational and of order $2(p+q)$ or $p+q$, according as q is odd or even.*

3. CARTESIAN AND HOMOGENEOUS EQUATIONS OF THE SURFACE.

Applying the formula

$$\tan rw = \frac{\binom{r}{1}\tan w - \binom{r}{3}\tan^3 w + \binom{r}{5}\tan^5 w - \cdots}{\binom{r}{0} - \binom{r}{2}\tan^2 w + \binom{r}{4}\tan^4 w - \cdots} \tag{8}$$

to the identity

$$\tan p\theta = \tan q \cdot \frac{p\theta}{q}, \tag{9}$$

we get

$$\frac{\binom{p}{1}\tan\theta - \binom{p}{3}\tan^3\theta + \binom{p}{5}\tan^5\theta - \cdots}{\binom{p}{0} - \binom{p}{2}\tan^2\theta + \binom{p}{4}\tan^4\theta - \binom{p}{6}\tan^6\theta + \cdots}$$

$$= \frac{\binom{q}{1}\tan\frac{p\theta}{q} - \binom{q}{3}\tan^3\frac{p\theta}{q} + \binom{q}{5}\tan^5\frac{p\theta}{q} - \cdots}{\binom{q}{0} - \binom{q}{2}\tan^2\frac{p\theta}{q} + \binom{q}{4}\tan^4\frac{p\theta}{q} - \binom{q}{6}\tan^6\frac{p\theta}{q} + \cdots}. \tag{10}$$

From (1)

$$\tan\theta = \frac{y}{x}, \qquad \tan\frac{p\theta}{q} = \frac{\rho - a}{z}.$$

Substituting this in (10), and assuming q odd, after some reduction, the cartesian equation of the surface is obtained in the form

$$\frac{\binom{p}{1}x^{p-1}y - \binom{p}{3}x^{p-3}y^3 + \binom{p}{5}x^{p-5}y^5 - \cdots}{\binom{p}{0}x^p - \binom{p}{2}x^{p-2}y^2 + \binom{p}{4}x^{p-4}y^4 - \cdots}$$

$$= \frac{\binom{q}{1}(\rho-a)z^{q-1} - \binom{q}{3}(\rho-a)^3z^{q-3} + \cdots + (-1)^{(q+1)/2}(\rho-a)^{q-2}z^2 + (-1)^{(q-1)/2}(\rho-a)^q}{\binom{q}{0}z^q - \binom{q}{2}(\rho-a)^2z^{q-2} + \cdots + (-1)^{(q+1)/2}(\rho-a)^{q-3}z^3 + (-1)^{(q-1)/2}(\rho-a)^{q-1}z}. \tag{11}$$

The expansions of $(\rho - a)^k$ bring in terms of the form $c_1(x^2+y^2)^m$, $c_2(x^2+y^2)^n\sqrt{x^2+y^2}$, where c_1 and c_2 are definite constants. Hence, to rationalize the equation, all terms with $\sqrt{x^2+y^2}$ as a factor must be collected on one side, all other terms on the other side of the equation. On squaring, a rational equation of order $2(p+q)$ is obtained, which is in agreement with the result of the foregoing section.

Monge* has shown that the equation of a ruled surface whose generatrices pass through the z-axis, has the form

$$z = x\psi\left(\frac{y}{x}\right) + \pi\left(\frac{y}{x}\right), \tag{12}$$

* "Application de l'Analyse à la Géométrie," 5 ed. (1850), pp. 83–89.

in which ψ and π are arbitrary functions (differentiable). This result he gained by differential geometry. Equation (11) can be readily put in the form (12). The left-hand member of (11) is a rational function of (y/x); the right-hand member a rational function of $(\rho - a)/z$. Consequently $(\rho - a)/z$ is an algebraic function of (y/x), say

$$(\rho - a)/z = R(y/x).$$

From this $\rho^2 = \{a + zR(y/x)\}^2$, or $x^2(1 + (y/x)^2) = \{a + zR(y/x)\}^2$. Extracting the square root, which we may assume positive, and solving for z, we have

$$z = x\frac{\sqrt{1 + (y/x)^2}}{R(y/x)} - \frac{a}{R(y/x)}, \tag{13}$$

so that Monge's proposition is verified for our class of surfaces.

Making use of the identities

$$\binom{r}{1}A^{r-1}B - \binom{r}{3}A^{r-3}B^3 + \binom{r}{5}A^{r-5}B^5 - \cdots = -\frac{i}{2}\{(A + iB)^r - (A - iB)^r\} \tag{14}$$

$$\binom{r}{0}A^r - \binom{r}{2}A^{r-2}B^2 + \binom{r}{4}A^{r-4}B^4 - \cdots = \frac{1}{2}\{(A + iB)^r + (A - iB)^r\} \tag{15}$$

and introducing the homogeneous isotropic coördinates $x_1 = x + iy$, $x_2 = x - iy$, $x_3 = as + iz$, $x_4 = as - iz$, where s denotes the variable making the system (x, y, z) homogeneous, equation (11) reduces to the symmetric irrational form

$$x_1x_2(x_1^{p/q} - x_2^{p/q})^2 - (x_3x_1^{p/q} - x_4x_2^{p/q})^2 = 0. \tag{16}$$

The intersection of this surface with the plane at infinity of the system x, y, z, s, is obtained by putting $s = 0$, so that $x_3 = iz, x_4 = -iz$. Applying the collineation $x_1/z = \xi$, $x_2/z = \eta$, the curve of intersection then assumes the form

$$\xi\eta(\xi^{p/q} - \eta^{p/q})^2 + (\xi^{p/q} + \eta^{p/q})^2 = 0.$$

where ξ and η are again cartesian coördinates. Introducing polar coordinates by putting $\xi = \rho e^{i\theta}$, $\eta = \rho e^{-i\theta}$, this reduces to the simple form

$$\rho = \tan\frac{p}{q}\theta. \tag{17}$$

From the cinematic definition of the surface it is apparent that this curve may be considered as the projection of the curve at infinity from the origin

upon the plane $z = 1$. That it is rational as proved before, is also apparent if we put $\theta/q = w$, then $\xi = \tan pw \cdot \cos qw$, $\eta = \tan pw \cdot \sin qw$ may be expressed rationally by the parameter $\tan(w/2)$.

To investigate the behavior of the curve at the isotropic points, we may write its equation in the rational projective form

$$(x_1^p + x_2^p)^2 x_1 x_2 \left\{ \binom{q}{1} x_3^{q-1} - \binom{q}{3} x_1 x_2 x_3^{q-3} + \cdots \right.$$
$$\left. \pm \binom{q}{4} x_1^{(q-5)/2} x_2^{(q-5)/2} x_3^4 \mp \binom{q}{2} x_1^{(q-3)/2} x_2^{(q-3)/2} x_3^2 \pm x_1^{(q-1)/2} x_2^{(q-1)/2} \right\}^2$$
$$+ (x_1^p - x_2^p)^2 x_3^2 \left\{ \binom{q}{0} x_3^{q-1} - \binom{q}{2} x_1 x_2 x_3^{q-3} + \cdots \right.$$
$$\left. \mp \binom{q}{3} x_1^{(q-3)/2} x_2^{(q-3)/2} x_3^2 \pm \binom{q}{1} x_1^{(q-1)/2} x_1^{(q-1)/2} \right\}^2,$$

in which the algebraic signs within the brackets are alternating throughout.

"Placing the curve on the analytic triangle," the terms nearest the vertex $X_1(x_2 = 0,\ x_3 = 0)$, after a rather tedious calculation, are found in order

$$x_1^{2p+q} x_2^q + \binom{q}{1} x_1^{2p+q-1} x_2^{q-1} x_3^2 + \binom{q}{2} x_1^{2p+q-2} x_2^{q-2} x_3^4 + \cdots + x_1^{2p} x_3^{2q} + \cdots,$$

and lie on a straight line of the "analytic triangle." Putting $x_1 = 1$. this may be written as

$$(x_2 + x_3^2)^q + \cdots,$$

so that in the neighborhood of the isotropic point $(x - iy = 0,\ z = 0)$ the curve has the same singularity as the curve $(x_2 + x_3^2)^q = 0$, which represents a q-fold parabola. This point must therefore be counted as $q(q-1)$ double points. As the equation is symmetric with respect to x_1 and x_2, also the other isotropic point $(x_1 = 0,\ x_3 = 0)$, or $(x + iy = 0,\ z = 0)$ must be counted as $q(q-1)$ double-points. The isotropic points must therefore be counted as $2q(q-1)$ double points.

When $q = 2s$ is even, the equation of the surface, which may also be written in the form

$$x_1^p (x_4 + i\rho)^q - x_2^p (x_3 - i\rho)^q = 0,$$

or

$$x_1^p \left[x_4^q - \binom{q}{2} x_4^{q-2} \rho^2 + \binom{q}{4} x_4^{q-4} \rho^4 - \cdots \right]$$
$$- x_2^p \left[x_3^q - \binom{q}{2} x_3^{q-2} \rho^2 + \binom{q}{4} x_3^{q-4} \rho^4 - \cdots \right]$$
$$= -i \left\{ x_1^p \left[\binom{q}{1} x_4^{q-1} \rho - \binom{q}{3} x_4^{q-3} \rho^3 + \cdots \right] \right.$$
$$\left. + x_2^p \left[\binom{q}{1} x_3^{q-1} \rho - \binom{q}{3} x_3^{q-3} \rho^3 + \cdots \right] \right\},$$

may be rationalized as follows:

Squaring both sides and making use of the identity

$$\left\{\binom{r}{0}A^r - \binom{r}{2}A^{r-2}B^2 + \binom{r}{4}A^{r-4}B^4 - \cdots\right\}^2 + \left\{\binom{r}{1}A^{r-1}B - \binom{r}{3}A^{r-3}B^3 + \binom{r}{5}A^{r-5}B^5 - \cdots\right\}^2 = (A^2+B^2)^r,$$

we get

$$x_1^{2p}(x_4^2+\rho^2)^q + x_2^{2p}(x_3^2+\rho^2)^q - 2x_1^p x_2^p\left\{\left[x_3^q - \binom{q}{2}x_3^{q-2}\rho^2 + \cdots\right]\left[x_4^q - \binom{q}{2}x_4^{q-2}\rho^2 + \cdots\right] - \left[\binom{q}{1}x_3^{q-1}\rho - \binom{q}{3}x_3^{q-3}\rho^3 + \cdots\right]\left[\binom{q}{1}x_4^{q-1}\rho - \binom{q}{3}x_4^{q-3}\rho^3 + \cdots\right]\right\} = 0. \quad (18)$$

The bracket-expression multiplying $2x_1^p x_2^p$ may be written in the form

$$\tfrac{1}{4}\big[\{(x_3+i\rho)^q + (x_3-i\rho)^q\}\{(x_4+i\rho)^q + (x_4-i\rho)^q\} + \{(x_3+i\rho)^q - (x_3-i\rho)^q\}\{(x_4+i\rho)^q - (x_4-i\rho)^q\}\big] = \tfrac{1}{2}\{(x_3+i\rho)^q(x_4+i\rho)^q + (x_3-i\rho)^q(x_4-i\rho)^q\}.$$

Then, substracting $2x_1^p x_2^p(x_3+\rho^2)^{q/2}(x_4^2+\rho^2)^{q/2}$ from the first two terms of (18), and adding the same expression to the third term, the equation may be written in the form

$$\{x_1^p(x_4^2+\rho^2)^{q/2} - x_2^p(x_3^2+\rho^2)^{q/2}\}^2 = x_1^p x_2^p\{(x_3+i\rho)^{q/2}(x_4+i\rho)^{q/2} - (x_3-i\rho)^{q/2}(x_4-i\rho)^{q/2}\}^2. \quad (19)$$

Now, since $q = 2s$, be expanding,

$$(x_3+i\rho)^s(x_4+i\rho)^s = x_3^s x_4^s + i\binom{s}{1}\rho\,(x_3^{s-1}x_4^s + x_3^s x_4^{s-1}) - i\rho^3\left[\binom{s}{1}\binom{s}{2}x_3^{s-1}x_4^{s-2} + \binom{s}{1}\binom{s}{2}x_3^{s-2}x_4^{s-1} + \binom{s}{3}x_3^s x_4^{s-3} + \binom{s}{3}x_3^{s-3}x_4^s\right] + \cdots.$$

The expansion of $(x_3-i\rho)^s(x_4-i\rho)^s$ is obtained from the foregoing by replacing i by $-i$. so that in the bracket-expression, multiplying $x_1^p x_2^p$ in (19), all terms without the factorf $\pm i$ cancel, leaving for it

$$2i\left\{\binom{s}{1}\rho(x_3^{s-1}x_4^s + x_3^s x_4^{s-1}) - \rho^3\left[\binom{s}{1}\binom{s}{2}(x_3^{s-1}x_4^{s-2} + x_3^{s-2}x_4^{s-1} + \binom{s}{3}(x_3^s x_4^{s-3} + x_3^{s-3}x_4^s)\right] + \cdots\right\}.$$

Taking out the factor ρ, and noting that $\rho^2 = x_1x_2$, (19) now assumes the form

$$\{x_1^p(x_4^2 + x_1x_2)^s - x_2^p(x_3^2 + x_1x_2)^s\}^2 = -4(x_1x_2)^{p+1}\left\{\binom{s}{1}(x_3^{s-1}x_4^s + x_3^s x_4^{s-1}) - x_1x_2\left[\binom{s}{1}\binom{s}{2}(x_3^{s-1}x_4^{s-2} + x_3^{s-2}x_4^{s-1}) + \binom{s}{3}(x_3^s x_4^{s-3} + x_3^{s-3}x_4^3)\right] + \cdots\right\}^2.$$

Extracting the square root on both sides of the equation which, according to geometric tests in examples given hereafter must be taken with the positive sign, we finally get for the equation of the surface, when q is even,

$$x_1^p(x_4^2 + x_1x_2)^s - x_2^p(x_3^2 + x_1x_2)^s - 2i(x_1x_2)^{p+1/2}\left\{\binom{s}{1}(x_3^{s-1}x_4^s + x_3^s x_4^{s-1}) - x_1x_2\left[\binom{s}{1}\binom{s}{2}(x_3^{s-1}x_4^{s-2} + x_3^{s-2}x_4^{s-1}) + \binom{s}{3}(x_3^s x_4^{s-3} + x_3^{s-3}x_4^s)\right] + \cdots\right\} = 0. \quad (20)$$

As p is odd, $p + 1$ is even, and $(p + 1)/2$ an integer, so that equation (20) is of degree $p + 2s = p + q$, and the surface consequently of order $p + q$, as was proved before in the parametric form. From this the cartesian form is easily obtained by substituting for x_1, x_2, x_3, x_4 their expressions in terms of x, y, z.

As in the case of q odd, the intersection of the surface with the plane at infinity can be placed upon the analytic triangle. By the same method is found that the isotropic points must be counted as $2s(s - 1)$ double points of the infinite curve of intersection in case of q even.

4. (α, β)-Correspondence between C_1 and C_2 Determined by the Generatrices of the Surface.

When we put $z = 0$ in (11), the intersection of the surface with the xy-plane is obtained

$$\left\{\binom{p}{0}x^p - \binom{p}{2}x^{p-2}y^2 + \binom{p}{4}x^{p-4}y^4 - \cdots\right\}(\rho - a)^q = 0. \quad (21)$$

To make this equation rational, put the bracket-expression equal to V, then $V^{1/q}(\rho - a) = 0$, $V^{2/q}(x^2 + y^2) = a^2V^{2/q}$, and from this as the equation of the intersection

$$V^2(x^2 + y^2 - a^2)^q = 0, \quad (22)$$

which shows that the directrix-circle C_2 is a q-fold curve of the surface,

when q is odd. The rest of the intersection consists of the p double generatrices in the xy-plane defined by $V^2 = 0$. These double lines divide the full angle into $2p$ equal parts, as is seen by actual determination of the angles from $V \equiv \rho^p(e^{ip\theta} + e^{-ip\theta}) = 0$, or $e^{2ip\theta} = -1$. From this is found $\theta = (k\pi/p) - (\pi/4p)$, so that θ has p values incongruent to π, determined by $k = 1, 2, 3, \cdots, p$. When q is odd, the midpoint M of the generatrix g describes C_2 q times, so that θ increases by $q \cdot 2\pi$ and $(p/q)\theta$ by $p \cdot 2\pi$. From this follows that g turns around p times in the plane e and sweeps $2p$ times through the z-axis, so that the z-axis or C_1 is a $2p$ fold line of the surface. To determine the positions of the generatrices in a plane e determined by an angle θ more definitely, let g_0 be the initial position of the generatrix determined by the angles θ and $\psi = (p/q)\theta$. As θ increases by π, e turns through an angle π and the generatrix moves to the position g_1 making an angle $(p/q)\theta + (p/q)\pi$ with the perpendicular to the xy-plane through the midpoint M_1 of g_1. As $\theta + \pi$ increases again by π, the generatrix g_1 moves to the new position g_2 through M, determined by the angle $(p/q)\theta + (p/q)2\pi$. While θ increases from 0 to $q \cdot 2\pi$, the generatrix describes the entire surface and returns point for point to its initial position g_0. Denoting the $2q$ positions of the generatrix by $g_0, g_1, g_2, \cdots, g_{2q-1}$ the corresponding determining angles may be listed in the table:

Through M		Through M_1	
g_0	$p/q\theta$	g_1	$p/q(\theta + \pi)$
g_2	$p/q(\theta + 2\pi)$	g_3	$p/q(\theta + 3\pi)$
g_4	$p/q(\theta + 4\pi)$	g_5	$p/q(\theta + 5\pi)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
g_{2q-2}	$p/q\{\theta + (2q-2)\pi\}$	g_{2q-3}	$p/q\{\theta + (2q-3)\pi\}$
g_{2q}	$p/q\{\theta + 2q\pi\}$	g_{2q-1}	$p/q\{\theta + (2q-1)\pi\}$

Through each M and M_1 there are q generatrices which divide the full angle around M as well as around M_1 in the plane e into $2q$ equal parts. Through every point of C_2 there are q generatrices.

To determine the number of generatrices through a point of C_1, let such a point be determined by $z =$ constant, then, $a\rho = 0$, from (1)

$$\cot \frac{p}{q}\theta = -\frac{z}{a} = \text{constant}. \tag{23}$$

From this $\frac{p}{q}\theta - k\pi = -\operatorname{arccot}\frac{z}{a}$, and

$$\theta = \frac{kq}{p}\pi - \frac{q}{p}\operatorname{arccot}\frac{z}{a}, \qquad \left(\operatorname{arccot}\frac{z}{a} \leqq \frac{\pi}{2}\right) \tag{24}$$

for $\theta \leqq \theta \leqq q \cdot 2\pi$. This gives for θ $2p$ values, determined by $k = 1, 2, 3, \cdots, 2p$, and incongruent $q \cdot 2\pi$, and consequently $2p$ generatrices through a point of the z-axis, or C_1. These lie in pairs of symmetric lines in planes through C_1, each pair being determined by two values k and k_1, such that their difference $k_1 - k = p$.

From the fact, that to every point of C_1 correspond $2p$ points of C_2, the points of intersection of the $2p$ generatrices through the point on C_1; and to every point of C_2 q points of C_1, the intersections of the q generatrices through the point on the C_2 with C_1, it is apparent that the generatrices cut out on C_1 and C_2 an (α, β)-correspondence of the specific form $(q, 2p)$.

To establish the algebraic form of this correspondence, we may represent C_2 parametrically by

$$x = a\frac{1 - \lambda^2}{1 + \lambda^2}, \qquad y = a\frac{2\lambda}{1 + \lambda^2},$$

so that conversely $\lambda = y/(a + x)$. If we now put $\tan(\theta/2q) = t$, then for $\rho = 0$,

$$z = -a \cot \frac{p}{q}\theta = -a \cot 2p \cdot \frac{\theta}{2q} = F(t), \tag{25}$$

$$\lambda = \frac{\sin\theta}{1 + \cos\theta} = \frac{2 \tan q \cdot \dfrac{\theta}{2q}}{1 - \tan^2 q \cdot \dfrac{\theta}{2q}} = G(t), \tag{26}$$

both rational functions of t. In $F(t)$ the highest power of t is $2p$, in $G(t)$ it is q. The elimination of t between (25) and (26) leads to a polynomial relation between z and λ which is of degree q in z, and of degree $2p$ in λ, and of degree $2p + q$ in both, and which is of deficiency 0. It has the form

$$\lambda^{2p}(a_0 z^q + a_1 z^{q-1} + \cdots) + \lambda^{2p-1}(b_0 z^q + b_1 z^{q-1} + \cdots) + \cdots + w_0 z^q + w_1 z^{q-1} + \cdots = 0. \tag{27}$$

This, when considered as a curve in a (λ, z)-plane, being of deficiency 0, has $\frac{1}{2}(2p + q - 2)(2p + q - 1)$ double-points, or double roots. But the infinite points of the λ and z-axis, are in the same order q and $2p$-fold points so that the number of finite double points is

$$\tfrac{1}{2}\{(2p + q - 2)(2p + q - 1) - 2p(2p - 1) - q(q - 1)\} = 2pq - 2p - q + 1. \tag{28}$$

But there are no real double generatrices which are simultaneously double lines of the system of $2p$ generatrices through a point of the z-axis and the q generatrices through a point of C_2. The surface has therefore a number of imaginary double-generatrices as given by (28). Hence

THEOREM 2. *When q is odd, the generatrices of the surface cut C_1 and C_2 in two point sets which are in a $(q, 2p)$-correspondence. C_1 and C_2 are $2p$-fold and q-fold curves of the surface. The surface has moreover p real and $2pq - 2p - q + 1$ imaginary double generatrices.*

When $q = 2s$ is even, $\psi = (p/2s)\theta$ increases from 0 to $p\pi$, when θ increases from 0 to $s \cdot 2\pi$. The midpoint M of the generatrix g turns s times about the z-axis, and g turns p times around M in the plane e. From this follows that C_1 and C_2 are respectively p- and s-fold lines of the surface. When θ increases from θ_0 to $\theta_0 + s \cdot 2\pi$, the generatrix g moves from the initial position g_0 to the position g_{2s} which coincides with g_0, but the segments into which M divides g_0 are interchanged on g_{2s}. In other words g describes a closed unifacial ruled surface.

From a point of C_1 there are p generatrices cutting C_2. This appears from the formula

$$\theta = \frac{kq}{p}\pi - \frac{q}{p}\operatorname{arccot}\frac{z}{a} \qquad \left(\operatorname{arccot}\frac{z}{a} \leqq \frac{\pi}{2}\right),$$

with $0 \leqq \theta \leqq s \cdot 2\pi$, which determines p values for θ by putting $k = 1, 2, 3, \cdots, p$. For two distinct values k and k_1 of this set we have $\theta_1 - \theta = [(k_1 - k)/p]q\pi$. As p is odd, q even, and $k_1 - k < p$, and p and q are relatively prime, $\theta_1 - \theta$ can never be an odd multiple of π. Hence no two of the generatrices through a point of C_1 can lie in the same plane e; hence the surface has no real double generatrices.

Through every point of C_1 there are p generatrices cutting C_2 in as many points; through every point of C_2 there are s generatrices cutting C_1 in the same number of points. Hence

THEOREM 3. *When $q = 2s$ is even the, generatrices of the surface cut C_1 and C_2 in two point sets which are in a rational (s, p)-correspondence. C_1 and C_2 are respectively p- and s-fold curves of the surface. The surface has no real, but has $ps - p - s + 1$ imaginary double generatrices.*

5. DOUBLE CURVES OF THE SURFACE WHEN q IS ODD.

The $2q$ generatrices in a plane e intersect in q^2 points D_{ik} of the double curve of the surface. Each pair of indices ik consists throughout of an odd and an even number. These q^2 couples may be arranged into $(q + 1)/2$ cyclic groups of which $(q - 1)/2$ are of order $2q$, and 1 is of order q. Those of order $2q$ are

01	03	05	$\cdots$	$0 \cdot q - 2$
12	14	16	$\cdots$	$1 \cdot q - 1$
23	25	27	$\cdots$	$2 \cdot q$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$2q - 1 \cdot 2q$	$2q - 1 \cdot 2$	$2q - 1 \cdot 4$	$\cdots$	$2q - 1 \cdot q - 3,$

that of order q is

$$\begin{matrix} 0q \\ 1\cdot q+1 \\ 2\cdot q+2 \\ \vdots \\ q-1\cdot 2q-1. \end{matrix}$$

In this group the generatrices g_0 and g_q, intersecting at D_{0q}, are determined by the angles $(p/q)\theta$ and $(p/q)(\theta+q\pi)=(p/q)\theta+p\pi$, which shows that when θ varies, the corresponding generatrices g_0 and g_q always intersect in a point of C_1, i.e., the intersections D of the group of order q all lie on C_1. On the other hand the intersections of each group of order $2q$ are cyclically permuted when θ increases by $q\cdot 2\pi$, so that the points D_{01}, D_{03}, D_{05}, $\cdots$ $D_{0\cdot q-1}$ describe $(q-1)/2$ twisted curves which together form the double curve of the ruled surface.

To find the equation of any of these double curves, for example the one described by the point starting from $D_{0\cdot 2k-1}$, where k may have any value between 1 and $(q-1)/2$, let x, y, z be the cartesian coördinates of $D_{0\cdot 2k-1}$, and ρ, θ the polar coördinates of x, y. The equations of g_0 and $g_{0\cdot 2k-1}$ in their planes (ρ, z) are

$$z=\quad(\rho-a)\cot\frac{p}{q}\theta, \qquad z=-(\rho-a)\cot\frac{p}{q}\{\theta+(2k-1)\pi\}.$$

From this

$$\rho=\frac{a\sin\frac{p}{q}(2k-1)\pi}{\sin\frac{p}{q}\{2\theta+(2k-1)\pi\}}, \tag{29}$$

which is the polar equation of the projection of the double curve upon the xy-plane. As $x=\rho\cos\theta$, $y=\rho\sin\theta$, the parametric (θ) equations of the double curve are

$$\begin{aligned} x&=\frac{a\sin\frac{p}{q}(2k-1)\pi\cdot\cos\theta}{\sin\frac{p}{q}\{2\theta+(2k-1)\pi\}}, \\ y&=\frac{a\sin\frac{p}{q}(2k-1)\pi\cdot\sin\theta}{\sin\frac{p}{q}\{2\theta+(2k-1)\pi\}}, \\ z&=-a\frac{\cos\frac{p}{q}\{2\theta+(2k-1)\pi\}+\cos\frac{p}{q}(2k-1)\pi}{\sin\frac{p}{q}\{2\theta+(2k-1)\pi\}}. \end{aligned} \tag{30}$$

From this all double curves are obtained by putting successively $k = 1, 2, 3, \cdots, (q-1)/2$.

Eliminating θ between (29) and the last equation of (30), the relation between ρ and z becomes

$$(\rho^2 - z^2 - a^2) \sin \frac{p}{q}(2k-1)\pi = 2\rho z \cos \frac{p}{q}(2k-1)\pi, \qquad (31)$$

which shows that the $2q$ intersections of the generatrices of a group in any plane e through the z-axis lie on two hyperbolas, which are symmetrical with respect to the z-axis, since to every increase of θ by π corresponds a rotation of the hyperbola (31) about the z-axis through an angle π. Thus the points $D_{01}, D_{23}, D_{45}, \cdots$ lie on (14), while $D_{12}, D_{34}, D_{56}, \cdots$ liè on the reflexion of (14). The double curve belonging to the group lies therefore on a surface of revolution of order 4 with the equation

$$(x^2+y^2-z^2-a^2)^2 \sin^2 \frac{p}{q}(2k-1)\pi - 4(x^2+y^2)z^2 \cos^2 \frac{p}{q}(2k-1)\pi = 0, \qquad (32)$$

with C_2 as a double curve.

The asymptotes of the hyperbola (31) are determined by the equation of their slope z/ρ,

$$\left(\frac{z}{\rho}\right)^2 + 2\left(\frac{z}{\rho}\right)\cot \frac{p}{q}(2k-1)\pi - 1 = 0,$$

which shows that the product of the slopes of the asymptotes is -1. (31) is therefore an equilateral hyperbola.

To find the generatrices of the surface with the same slopes, in other words the infinite points of the double curve described by $D_{0\cdot 2k-1}$, the conditions must be satisfied:

(a) $$\cot \frac{p}{q}\theta = \tan \frac{p(2k-1)\pi}{2q},$$

and from this

$$\theta = \frac{q(2l+1)}{2p}\pi - \frac{(2k-1)\pi}{2},$$

for all values $l = 0, 1, 2, \cdots, 2p-1$. Two consecutive values of θ differ by the amount $(q/2p)\pi$. For every value of θ of this set there is a value $\theta' = \theta + q\pi$. For a definite value of l the angles are

$$\theta' = \frac{q\{2(l+p)+1\}}{2p}\pi - \frac{2k-1}{2}\pi \text{ and } \theta = \frac{q(2l+1)}{2p}\pi - \frac{2k-1}{2}\pi.$$

Hence there are $2p$ values of θ due to condition (1) for which the double curve has infinite points. The condition

(b) $$\cot \frac{p}{q}\theta = -\cot \frac{p}{2q}(2k-1)\pi$$

is satisfied, when

$$\theta = \frac{ql\pi}{p} - \frac{2k-1}{2}\pi,$$

which again determines $2p$ values for θ, for $l = 1, 2, 3, \cdots, 2p-1$. The values of θ determined by (a) and (b) are also obtained from the condition that in (29) $\rho = \infty$. This is the case for $(p/q)\{2\theta + (2k-1)\pi\} = m\pi$, or

$$\theta = \frac{mq\pi}{2p} - \frac{(2k-1)\pi}{2},$$

for $m = 1, 2, 3, \cdots, 4p$, and $k \leqq (q-1)/2$. For $k = (q+1)/2$ and $m = 1$, we get $\theta = 0$ and $\rho = a$. This corresponds to the one group of order q.

The order of the double curve may be determined by expressing sines and cosines in (30) by tangents. In the first place let $\theta + [(2k-1)/2]\pi = \phi$, or $\theta = -\{(2k-1)(\pi/2) - \phi\}$, so that $\sin\theta = (-1)^k \cos\phi$, $\cos\theta = (-1)^{k+1} \sin\phi$. Then (30) may be written in the form

$$\begin{aligned} x &= a \cdot \frac{(-1)^{k+1} \sin\frac{p}{q}(2k-1)\pi \cdot \sin\phi}{\sin\frac{p}{q}2\phi}, \\ y &= a \cdot \frac{(-1)^{k} \sin\frac{p}{q}(2k-1)\pi \cos\phi}{\sin\frac{p}{q}2\phi}, \\ z &= -a \cdot \frac{\cos\frac{p}{q}2\phi + \cos\frac{p}{q}(2k-1)\pi}{\sin\frac{p}{q}2\phi}. \end{aligned} \tag{33}$$

Expressing sines and cosines in terms of $\tan(\theta/2q) = t$ by means of identities as given in (9), and using similar notations by functions ϕ, ψ, ϕ_1, ψ_1, of t the parametric equations of the double curve become

$$\begin{aligned} x &= (-1)^{k+1} a \sin\frac{p}{q}(2k-1)\pi \frac{\phi^q(t)\psi^{q-1}(t)(1+t^2)^{2p-q}}{\phi^{2p}(t)\cdot\psi^{2p-1}(t)}, \\ y &= (-1)^{k} a \sin\frac{p}{q}(2k-1)\pi \frac{\{\psi^{2q-1}(t) - \phi^{2q}(t)\}(1+t^2)^{2p-q}}{2\phi^{2p}(t)\cdot\psi^{2p-1}(t)}, \\ z &= -a \frac{\psi^{4p-2}(t) - \phi^{4p}(t) + \cos\frac{q}{p}(2k-1)\pi\cdot(1+t^2)^{2p}}{2\phi^{2p}(t)\cdot\psi^{2p-1}(t)}, \end{aligned} \tag{34}$$

which are rational functions of t. The highest power of t in these functions is $4p$, or $2q$, according as $q \leqq 2p$. Hence

THEOREM 4. *The order of each of the* $(q-1)/2$ *double curves is* $4p$ *or* $2q$ *according as* $q \lesseqgtr 2p$, *and when* q *is odd. They are rational and each lies on a surface of revolution of order* 4 *generated by the rotation of an equilateral hyperbola about the* z*-axis.*

As the surfaces of the class are rational, all plain sections are rational and must therefore have the maximum number of double points. We shall verify this number in case of q odd, and $q \leqq 2p$, for the curve of intersection with the plane at infinity. Consider first a plane $z = \text{const.}$, cutting the z-axis in a point Z. A pair of generatrices g and g' in the same plane e and through the same point U (different from Z) of the z-axis cut the plane $z = \text{const.}$ in two points A and A', which are collinear with Z. Now as θ increases, A and A' describe curves in central symmetry with Z, which will touch each other at Z, i.e., the form or tac-nod at Z. But Z must be counted as a $2p$-fold point of the curve of intersection with the plane $z = \text{const.}$, since the z-axis is a $2p$-fold line of the surface, and as it occurs p times that two branches of the curve form tac-nods, the point Z must be counted as

$$\frac{2p(2p-1)}{2} + p = 2p^2$$

double points. As this is true for every point of the z-axis, the latter must be counted as $2p^2$ double lines. The surface has therefor $2p^2 + p$ real double lines (p double lines in the xy-plane. The double points of the infinite curve of intersection are therefore made up of those absorbed by the isotropic points, of the intersections with the real double lines, with the imaginary double-lines, and with the $(q-1)/2$ double curves, each with $4p$ infinite points. These numbers are in the same order, and summed up:

$$\{2q(q-1)\} + \{2p^2 + p\} + \{2pq - 2p - q + 1\} + \left\{\frac{q-1}{2} \cdot 4p\right\} = \frac{\{2(p+q)-1\}\{2(p+q)-2\}}{2},$$

as required of a plain curve of order $2(p+q)$.

6. DOUBLE CURVES OF THE SURFACE WHEN g IS EVEN.

When $q = 2s$, there are two pencils, of s generatrices each, in every plane section through the z-axis, which intersect in s^2 points. When the plane e turns s times about the z-axis, starting from an initial position e_0 containing the initial position g_0 of the generatrix g, assuming in succession positions determined by the angles $\theta = 0, \pi, 2\pi, 3\pi, \cdots$, g will occupy the positions $g_0, g_1, g_2, \cdots, g_{2s-1}$. In this movement the intersection of the generatrices of even and odd indices in the same plane describe double curves of the surface. Every double curve, with the exception of one in

case when s is odd, cuts the initial plane e_0 in $2s$ points. When s is odd the number of intersections of the one curve with the plane is s, and as

$$s^2 = \frac{s-1}{2} \cdot 2s + s,$$

there are in this case $(s-1)/2$ double curves with $2s$ intersections, and one curve with s intersections with the plane e_0; i.e., altogether $(s+1)/2$ double curves. To each of these curves corresponds a cyclic group of points of intersection with the plane e_0. Denoting the groups of order $2s$ by $D_1, D_2, D_3, \cdots, D_{(s-1)/2}$ and the group of order s by $D_{(s-1)/2}$ the following table of these groups may be set up:

D_1	D_2	D_3	$\cdots$	$D_{(s-1)/2}$	$D_{(+1s)/2}$
01	03	05	$\cdots$	$0\cdot(s-2)$	$0\cdot(s)$
12	14	16	$\cdots$	$1\cdot(s-1)$	$1\cdot(s+1)$
23	25	27	$\cdots$	$2\cdot(s+1)$	$2\cdot(s+2)$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$(2s-2)\cdot(2s-1)$	$(2s-2)\cdot 1$	$(2s-2)\cdot 3$	$\cdots$	$(2s-2)\cdot(s-4)$	$(s-1)\cdot[2(s-1)+1]$
$(2s-1)\cdot 0$	$(2s-1)\cdot 2$	$(2s-1)\cdot 4$	$\cdots$	$(2s-1)\cdot(s-3)$	

When $s = 2\sigma$ is even, then

$$s^2 = 4\sigma^2 = \sigma\cdot 4\sigma = \sigma\cdot 2s.$$

There are σ double curves, each with $2s$ intersections with the initial plane. The table of groups is in this case:

D_1	D_2	D_3	$\cdots$	D_σ
01	03	05	$\cdots$	$0\cdot(s-1)$
12	14	16	$\cdots$	$1\cdot(s)$
23	25	27	$\cdots$	$2\cdot(s+1)$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$(2s-2)\cdot(2s-1)$	$(2s-2)\cdot 1$	$(2s-2)\cdot 3$	$\cdots$	$(2s-2)\cdot(s-3)$
$(2s-1)\cdot 0$	$(2s-1)\cdot 2$	$(2s-1)\cdot 4$	$\cdots$	$(2s-1)\cdot(s-2)$

The parametric equations of the double curves are obtained from those of (33) by putting $\tan(\phi/2s) = t$, so that in analogy with previous results

$$\sin\phi = \frac{2\tan s\cdot\frac{\phi}{2s}}{1+\tan^2 s\cdot\frac{\phi}{2s}} = \frac{2\phi^s(t)\cdot\psi^{s-1}(t)}{(1+t^2)^s},$$

$$\cos\phi = \frac{1 - \tan^2 s\cdot\dfrac{\phi}{2s}}{1 + \tan^2 s\cdot\dfrac{\phi}{2s}} = \frac{\psi^{2(s-1)}(t) - \phi^{2s}(t)}{(1+t^2)^s},$$

$$\cos\frac{p}{s}\phi = \frac{1 - \tan^2 p\cdot\dfrac{\phi}{2s}}{1 + \tan^2 p\cdot\dfrac{\phi}{2s}} = \frac{\phi_1^{2(p-1)}(t) - \psi_1^{2p}(t)}{(1+t^2)^p},$$

and the parametric equations of the double curve are

$$x = (-1)^{k+1}\, a \sin\frac{p}{2s}(2k-1)\pi \frac{\phi^s(t)\psi^{s-1}(t)(1+t^2)^{p-s}}{\phi_1^p(t)\cdot\psi_1^{p-1}(t)},$$

$$y = (-1)^k\, a \sin\frac{p}{2s}(2k-1)\pi \frac{\{\psi^{2(s-1)}(t) - \phi^{2s}(t)\}(1+t^2)^{p-s}}{2\phi_1^p(t)\cdot\psi_1^{p-1}(t)}, \qquad (35)$$

$$z = -\,a\,\frac{\phi_1^{2(p-1)}(t) - \psi_1^{2p}(t) + \cos\dfrac{p}{2s}(2k-1)\pi\cdot(1+t^2)^{p-s}}{2\phi_1^p(t)\cdot\psi_1^{p-1}(t)}.$$

When $p > s$, or $q < 2p$, and $2k - 1 \neq s$, then the highest power of t in these expressions is t^{2p}, so that in this case the double curves are of order $2p$. When $p < s$, then they are of order $2s = q$.

In case of the double curve of order s, which is obtained from (33) by putting $2k - 1 = s$, when s is odd, the coördinates may be expressed parametrically by $t = \tan(\phi/s)$ as rational functions of t, in which the highest power of t is t^p or t^s, according as $p > s$, or $p < s$, so that the double curve $[D(s+1)/2]$ is accordingly of order p or s, as appears from the parametric equations of the surface

$$x = \pm\frac{(1+t^2)^{(p-s)/2}\left[\binom{s}{1} - \binom{s}{3}(1+t^2)t^2 + \cdots \pm t^{s-1}\right]}{\binom{p}{1} - \binom{p}{3}(1+t^2)t^2 + \cdots \pm t^{p-1}},$$

$$y = \pm\frac{(1+t^2)^{(p-s)/2}\left[1 - \binom{s}{2}t^2 + \cdots \pm t^{s-1}\right]}{\binom{p}{1}t - \binom{p}{3}t^3 + \cdots \pm t^p}, \qquad (36)$$

$$z = -\,a\cot\left(\frac{p}{s}\phi\right) = f(t),$$

where the signs within the brackets and in the denominators are alternating throughout, and $f(t)$ denotes a rational function in which the highest power of t is t^p. Hence

THEOREM 5. *When $q = 2s$ is even and s odd, there are $(s-1)/2$ double curves of order $2p$ or q according as $p > s$ or $p < s$, and one double curve of order p or s, according as $p > s$ or $p < s$. When $s = 2\sigma$ is even, there are σ double curves of order $2p$ or q, according as $p > s$ or $p < s$.*

7. APPLICABILITY AMONG THE RULED SURFACES OF THE CLASS.*

Consider first the case of an odd q. Let the generatrix g whose midpoint M moves along the directrix circle C_2 generate the surface determined by the numbers p and q. At every position of M draw in the xy-plane an external tangent circle C_2' to C_2 with the radius $(m/n)a$. Through the center $0'$ of C_2' draw a perpendicular z' to the plane of C_2', and let g in every position be asscoiated with C_2' and z', just as it is associated with C_2 and z. Thus, the generatrices g associated with C_2', z' will generate a ruled surface F' whose directrix circle is C_2' and whose directrix line is z'. To determine the nature of this surface, we must determine how many revolutions C_2' has to make about the axes z' and z respectively, before the initial point M_0 on C_2', after a certain number of revolutions about the z'-axis, and the initial generatrix g_0 associated with F', after rotating a certain number of times about M in the plane e', return to the initial positions of M on C_2, and g_0 associated with the given surface. This will be the case when a certain multiple ν of the circumference of C_2' is equal to a certain multiple of q times the circumference of C_2, i.e., $\nu \cdot 2(m/n)a\pi = \mu q \cdot 2a\pi$. This gives for μ/ν the ratio

$$\frac{\mu}{\nu} = \frac{m}{nq}.$$

Assuming m and n, also m and q as relative primes, we may put $\mu = m$, $\nu = nq$. From this is seen that the ruled surface F' is generated by a generatrix whose midpoint describes C_2' nq times. The surface F' turns mq times about the z-axis. Moreover g associated with F' makes mp complete revolutions in the plane e' through the z'-axis. The order of the surface is therefore $2(mp + nq)$. Hence

THEOREM 6. *Surfaces of the class are applicable to each other when their orders are $2(p + q)$ and $2(mp + nq)$, and their radii of C_2 respectively a and $(m/n)a$, when q is odd, p and q, m and n, and m and q are relative primes.* This is still true when either $p = q = 1$, or $m = n = 1$, hence

Corollary 7. *A surface of the class of order $2(p + q)$, q odd, is applicable upon a surface of the class of order* 4.

Corollary 8. *A surface of the class of order $2(p + q)$, q odd, is applicable to another surface of the same class and order.*

* See Eisenhart, loc. cit., pp. 342–347.

When q is even, $q = 2s$, then we have under similar conditions

$$\frac{\mu}{\nu} = \frac{m}{ns},$$

and $mp + nq$ as the order of the surface. As m and p are odd, and q is even, $mp + nq$ is an odd number, so that we may state

THEOREM 9. *Surfaces of the class of odd order are applicable to each other when their orders are $p + q$ and $mp + nq$, and their radii a and $(m/n)a$, respectively, and when q is even and p and m are odd.*

As the surfaces of even and odd order are bifacial and unifacial respectively, we have

Corollary 10. *Bifacial and unifacial surfaces of the class are applicable to surfaces of the same type only.*

Similar results would be obtained by choosing for C_2' an internal tangent circle of C_2.

8. INTERSECTIONS OF THE SURFACES OF THE CLASS WITH A TORUS.

A torus whose circular axis coincides with the directrix circle C_2 of the surface cuts the surface in a composite curve of order $8(p + q)$ or $4(p + q)$, according as q is odd or even. If the radius of a meridian cross-section of the torus is b, then it follows easily that a part of this composite curve, generated by a point P on the generatrix g whose distance $PM = b$ from M is constant may be represented parametrically by

$$\begin{aligned} x &= \left(a + b \sin \frac{p}{q}\theta\right) \cos\theta, \\ y &= \left(a + b \sin \frac{p}{q}\theta\right) \sin\theta, \\ z &= b \cos \frac{p}{q}\theta. \end{aligned} \tag{37}$$

Putting again $\tan(\theta/2q) = t$, x, y, z may be expressed as rational functions of t, in which the highest power of t is $t^{2(p+q)}$, so that the order of the curve is $2(p + q)$. The polar equation of the projection of the curve upon the xy-plane is

$$\rho = a + b \sin \frac{p}{q}\theta, \tag{38}$$

which represents a socalled cyclo-harmonic curve,* which, in general, is of order $2(p + q)$, and is rational.

* R. E. Moritz, "On the Construction of Certain Curves Given in Polar Coördinates, *The American Mathematical Monthly*, Vol. XXIV, pp. 213–220 (1917).

When $a = 0$, and one of the integers p, q is even, the other odd, the curve is still of order $2(p + q)$; but when both are odd, then, as is well known,* the curve is of order $p + q$.

Expanding the identity $\sin p\theta = \sin q \cdot (p\theta/q)$, and substituting

$$\sin\frac{p}{q} = \frac{\rho - a}{b}, \qquad \sin\theta = \frac{y}{\rho}, \qquad \cos\theta = \frac{x}{\rho}, \qquad \cos\frac{p}{q}\theta = \frac{\sqrt{b^2 - (\rho - a)^2}}{b},$$

the cartesian equation of the curve (38) may be written in the form

$$b^q\left\{\binom{p}{1}x^{p-1}y - \binom{p}{3}x^{p-3}y^3 + \binom{p}{5}x^{p-5}y^5 - \cdots\right\}$$
$$= \rho^p\left\{\binom{q}{1}\left[b^2 - (\rho - a)^2\right]^{(q-1)/2}(\rho - a) - \binom{q}{3}\left[b^2 - (\rho - a)^2\right]^{(q-3)/2}(\rho - a)^3 + \cdots\right\}. \quad (39)$$

9. EXAMPLES.

A. Bifacial Surfaces.

1. *Quartic,*† $p = 1$, $q = 1$.

$$(x^2 + y^2)x^2 - (ax + yz)^2 = 0.$$

C_2 is a single curve, C_1 must be counted as two double lines. The y-axis is the only double line in the xy-plane.

2. *Sextic,* $p = 2$, $q = 1$.

$$(x^2 + y^2)(x^2 - y^2)^2 - \{2xyz - a(x^2 - y^2)\}^2 = 0.$$

C_2 is a single curve, C_1 must be counted as eight double lines, and there are two double lines in the xy-plane with the equations $x \pm y = 0$.

B. Unifacial Surfaces.

3. *Cubic,* $p = 1$, $q = 2$

$$\{(x - a)^2 - (z - y)^2\}y + 2x(x - a)(z - y) = 0,$$

or in parametric form

$$x = \rho\frac{1 - t^2}{1 + t^2}, \qquad y = \rho\frac{2t}{1 + t^2}, \qquad z = \frac{\rho - a}{t}.$$

The one double curve consists of the straight line $x = a$, $y = at$, $z = at$. Two points P_1 and P_2 on the generatrix g equally distant from M bound a

* Gino Loria, "Spezielle algebraische und transzendente Kurven," 2d ed., Vol. 1, pp. 358–369.

† This is the sixth species of the classification of quartic scrolls by Cayley, Collected Works, Vol. 6, p. 328, and the fifth species of that of Cremona in *Memorie della Accademia delle Scienze dell' Istituto di Bologna*, 2 Ser., Vol. 8, pp. 235–250 (1868).

segment which describes a unifacial band of Moebius.* In a similar manner, and in a more general sense, such a band is described on any surface of the class, and is unit or bifacial according as the order of the surface is odd or even. This surface is also discussed from a function theoretic standpoint by Weyl†.

4. *Quintic*, $p = 3$, $q = 2$.

$$(y^3 - 3x^2y)z^2 + (2x^4 + 4x^2y^2 + 2y^4 - 2ax^3 + 6axy^2)z + (x^2 + y^2 - a^2)(y^3 - 3x^2y) = 0.$$

C_1 is a triple line of the surface, C_2 is single. There is one double curve of order 3 with the parametric equations

$$x = \frac{a(1 + t^2)}{1 - 3t^2}, \qquad y = \frac{at(1 + t^2)}{1 - 3t^2}, \qquad z = \frac{a(3t - t^3)}{1 - 3t^2}.$$

The projection of this curve upon the xy-plane has the polar equation

$$\rho = \frac{a}{\cos 3\theta},$$

and the cartesian equation

$$x^3 - 3xy^2 - a(x^2 + y^2) = 0.$$

This quintic has the index AIII in Schwarz's classification of quintic ruled surfaces, loc. cit., p. 57.

C. Number of Surfaces of a given Order.

When the order of the surface of the class is given, we may ask the question, how many species of the class are there with the same order? When the order is even, say $2n$, then $p + q = n$, and we may form all possible combinations of relative primes p and q with q odd, to satisfy this condition. For example when the order is 12, $p + q = 6$, and we have the possibilities (1) $p = 5$, $q = 1$; (2) $p = 2$, $q = 3$; (3) $p = 1$, $q = 5$, so that there are three surfaces of order 12. When the order is odd, say $p + q = m$, we may proceed in a similar manner; for example when $m = 13$. As q is now even we have the possibilities (1) $p = 11$, $q = 2$; (2) $p = 9$, $q = 4$; (3) $p = 7$, $q = 6$; (4) $p = 5$, $q = 8$; (5) $p = 3$, $q = 10$; (6) $p = 1$, $q = 12$. Hence there are six surfaces of the class of order 13.

In general, when the order is even, say $2n$ and n is even, then there are at most $n/2$ species of the same order; when n is odd, there at most $(n + 1)/2$. When the order is odd, say m, there are at most $(m - 1)/2$ species of the class with this order.

* Werke, Vol. 2, pp. 484–485 and pp. 519–521.

† "Die Idee der riemannschen Fläche," p. 26.